Approximation of the Dispersionless Toda Lattice by Toeplitz Operators

Matthew Thomas

May 14, 2009

Overview

- system of particles with a nearest-neighbor interaction - The Toda lattice
- potential function of the form $V(r) = \frac{r^2}{2}$
- 1955 - Enrico Fermi, John Pasta, and Stanislaw Ulam conducted experiments to understand a 1-dimensional crystal
- Morikazu Toda investigated a variation using potential $V(r) = e^{-r} + r - 1$
- Toda's system has soliton solutions - solutions which do not change shape or size over time

Overview

- overview of Hamiltonian mechanics
- symplectic geometry required to do Hamiltonian mechanics
- the Toda lattice
- geometric quantization
- geometry - connections, curvature, line bundles, and cohomology
- Toeplitz operators constructed for approximation
- dispersion
- sketch of the ideas of Bloch, Golse, Paul, and Uribe

Hamiltonian Mechanics

Definition

A *Hamiltonian function* is a smooth function

$$\mathcal{H} : (q^1, q^2, \dots, q^n, p_1, p_2, \dots, p_n) \to \mathbb{R},$$

- the q^i coordinate represents the position of the i^{th} mass, and the p_i coordinate represents the momentum of the i^{th} mass.
- Hamiltonian function often represents the energy of a system.

Definition (Hamiltonian vector field)

For a given Hamiltonian function $\mathcal{H}$, the *Hamiltonian vector field* is defined by

$$\mathcal{X}_{\mathcal{H}} = \frac{\partial \mathcal{H}}{\partial p_i} dq^i - \frac{\partial \mathcal{H}}{\partial q^i} dp_i = \left(\frac{\partial \mathcal{H}}{\partial p_1}, \dots \frac{\partial \mathcal{H}}{\partial p_n}, -\frac{\partial \mathcal{H}}{\partial q^1}, \dots, -\frac{\partial \mathcal{H}}{\partial q^n} \right)$$

- the set of points $(q^1, q^2, \dots, q^N, p_1, p_2, \dots, p_N)$ is called the *phase space* for a classical mechanical system
- the set of points $(q^1, q^2, \dots, q^n)$ is called the *configuration space* for a classical mechanical system
- an integral path for the Hamiltonian vector field determines the path followed in time by the system

time derivatives of position and momentum can be computed:

Definition (Hamiltonian equations)

Hamiltonian equations

$$\begin{cases} \dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i}. \end{cases}$$

Particle in a Potential field

- a single particle with potential function $V(q)$ has a Hamiltonian of the form

$$H(q,p) = \frac{p^2}{2m} + V(q).$$

- Hamiltonian equations are

$$\begin{cases} \dot{q} & = \frac{p}{m} \\ \dot{p} & = -\nabla V. \end{cases}$$

- these are Newton's laws in Hamiltonian mechanics.

Poisson bracket

Definition

Given two functions f and g (each of q^i and p_i), the *Poisson bracket* of f and g is given by

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}.$$

- allows the time derivative of an arbitrary function F to be computed along solution curves of the Hamiltonian vector field by

$$\dot{F} = \{F, \mathcal{H}\}$$

Why?

$$\dot{F} = \{F, \mathcal{H}\}$$

because

$$\frac{dF}{dt} = \sum_i \left(\frac{\partial F}{\partial q^i} \dot{q}^i + \frac{\partial F}{\partial p_i} \dot{p}_i \right) = \sum_i \left(\frac{\partial F}{\partial q^i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q^i} \right) = \{F, \mathcal{H}\}.$$

The Poisson bracket is antisymmetric, so in particular,

$$\dot{\mathcal{H}} = \{\mathcal{H}, \mathcal{H}\} = 0.$$

Therefore, the energy of the system is conserved.

The Toda Lattice

- N identical masses connected by identical springs in a 1-dimensional chain
- units chosen to normalize the mass $m = 1$ and the resistance of the system $k = 1$

$$\mathcal{H}(q^1, q^2, \ldots, q^N, p_1, p_2, \ldots, p_N) = \frac{1}{2}\sum_{j=1}^{N} p_i^2 + \sum_{j=1}^{N-1} e^{q^j - q^{j+1}}$$

- kinetic energy is $\frac{1}{2}\sum_{j=1}^{N} p_i^2$
- potential energy is $\sum_{j=1}^{N-1} e^{q^j - q^{j+1}}$

The associated Hamiltonian equations are

$$
\begin{cases}
\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i} = p_i, \ \ i = 1, \ldots, N \\
\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i} = e^{q^{i-1} - q^i} - e^{q^i - q^{i+1}}, \ \ i = 2, \ldots, N-1 \\
\dot{p}_1 = -\frac{\partial \mathcal{H}}{\partial q^1} = -e^{q^1 - q^2} \\
\dot{p}_N = -\frac{\partial \mathcal{H}}{\partial q^N} = e^{q^{N-1} - q^N}
\end{cases}
$$

Another construction for the Toda lattice

Let the potential be $V(r) = e^{-r} + r - 1$
Hamiltonian function is:

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i=1}^{N-1} V(q^{i+1} - q^i)$$

Why would this be a good potential?

$V(r) = e^{-r} + r - 1$ has Taylor series approximation $V(r) \approx \frac{r^2}{2}$ (harmonic potential)

The equations of motion for this Hamiltonian are:

$$\begin{cases} \dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i} = p_i \text{ for } i = 1, \ldots, N \\ \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i} = e^{q^{i-1}-q^i} - e^{q^i - q^{i+1}} \text{ for } i = 2, \ldots, N-1 \\ \dot{p}_1 = -\frac{\partial \mathcal{H}}{\partial q^1} = 1 - e^{q^1 - q^2} \\ \dot{p}_N = -\frac{\partial \mathcal{H}}{\partial q^N} = -1 + e^{q^{N-1} - q^N} \end{cases}$$

- nearly the same as those of the original definition of the Hamiltonian
- we will use the original Hamiltonian function

Change of coordinates

By applying the change of coordinates

$$\begin{cases} a_j = \frac{1}{2} e^{\frac{q^j - q^{j+1}}{2}} \\ b_j = -\frac{p_j}{2}, \end{cases}$$

the following relations hold:

$$\dot{a}_n = a_n(b_{n+1} - b_n)$$
$$\dot{b}_n = 2(a_n^2 - a_{n-1}^2)$$

Boundary Values

- indeterminacy in the change of coordinates
- to determine solutions to the differential equations, we need to know the values of a_0 and a_N
- choosing q^0 and q^{N+1} $\leftrightarrow$ choosing a_0 and a_N
- $a_0 = a_N = 0$ - called the *non-periodic Toda lattice*
- corresponds to formally setting $q_0 = -\infty$ and $q_{N+1} = +\infty$
- $a_{j+N} = a_j$ and $b_{j+N} = b_j$ - called the *periodic Toda lattice*.
- can define a quasi-periodic boundary condition for the Toda lattice by $a_N = e^{-2\pi\nu} a_0$

Matrix Description of the Toda Lattice

Definition

A *Lax pair* is a pair of matrices $L(t)$ and $B(L(t))$ such that

$$\frac{dL}{dt} = [B(L(t)), L(t)].$$

- The change of coordinates into the a and b allow the equations of motion to be written as a Lax pair

The Periodic Case

$$L(t) = \begin{pmatrix} b_1 & a_1 & 0 & \dots & a_N \\ a_1 & b_2 & a_2 & \dots & 0 \\ 0 & a_2 & b_3 & a_3 \dots & 0 \\ \dots & \dots & & \dots & \vdots \\ a_N & \dots & & a_{N-1} & b_N \end{pmatrix}$$

and

$$B(L(t)) = \begin{pmatrix} 0 & a_1 & 0 & \dots & -a_N \\ -a_1 & 0 & a_2 & \dots & 0 \\ 0 & -a_2 & 0 & a_3 \dots & 0 \\ \dots & \dots & & \dots & \vdots \\ a_N & \dots & & -a_{N-1} & 0 \end{pmatrix}$$

Non-periodic case

$$L(t) = \begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 \\ a_1 & b_2 & a_2 & \dots & 0 \\ 0 & a_2 & b_3 & a_3 \dots & 0 \\ \dots & \dots & & \dots & \vdots \\ 0 & \dots & & a_{N-1} & b_N \end{pmatrix}$$

and

$$B(L(t)) = \begin{pmatrix} 0 & a_1 & 0 & \dots & 0 \\ -a_1 & 0 & a_2 & \dots & 0 \\ 0 & -a_2 & 0 & a_3 \dots & 0 \\ \dots & \dots & & \dots & \vdots \\ 0 & \dots & & -a_{N-1} & 0 \end{pmatrix}$$

Definition

An *integral of motion* is a function f such that $\{f, H\} = 0$.

Definition

If $\mathcal{H}(q^1, q^2, \ldots, q^n, p_1, p_2, \ldots, p_n)$ is a Hamiltonian function for a system, then the Hamiltonian system is called *integrable* or *completely integrable* if there exist n integrals of motion I_j such that $\{I_j, I_k\} = 0 \; \forall j \neq k$ and $dI_1 \wedge dI_2 \wedge \ldots \wedge dI_n \neq 0$.

- the Hamiltonian function will be an integral of motion
- for the Toda lattice, the eigenvalues of the matrix L (assumed distinct) are integrals of motion since $\frac{d\lambda}{dt} = 0$ by Lax pair formulation
- The periodic and non-periodic Toda lattices are both integrable systems

First Discussion of The Result of Bloch, Golse, Paul, and Uribe

- Consider two continuous functions on the unit interval, a and b
- We will fixed particular values of these functions by demanding that they satisfy conditions related to the Toda lattice

Let

$$\begin{cases} a(\frac{j}{N}) = a_j \\ b(\frac{j}{N}) = b_j. \end{cases}$$

- The spacing between the fixed values is $\frac{1}{N}$. If we then allow the functions a and b to vary over time, t, we can then require that a_j^t and b_j^t satisfy the equations of motion $\dot{a}_n = a_n(b_{n+1} - b_n)$ and $\dot{b}_n = 2(a_n^2 - a_{n-1}^2)$

Definition

Define the variables

$$\begin{cases} x := \frac{j}{N} \\ s := \frac{t}{N}. \end{cases}$$

Proposition

Let a_j^t and ${}_j^t$ be defined as above. These relations, under the change of variables above and in the limit as $N \to \infty$ produce the partial differential equations

$$\begin{cases} \partial_s a^s(x) = a^s(x) \partial_x b^s(x) \\ \partial_s b^s(x) = 2\partial_x (a^s(x))^2. \end{cases}$$

This is called the *dispersionless limit of the Toda lattice*.

Proposition

This system is hyperbolic.

Hyperbolicity implies the system may develop shocks, depending on the initial conditions. This can be seen by imposing the initial conditions $b^{t=0} = 2a^{t=0}$, which reduces to Burger's equation:

$$\partial_s a = \partial_x a^2,$$

which is known to develop shocks.

Symplectic Geometry

Definition

A *symplectic manifold*, (M, ω) is a smooth manifold, M, equipped with a nondegenerate skew-symmetric, closed 2-form, ω, called the *symplectic form*.

Can be written in local coordinates:

- ω will be written as a matrix at each point in the manifold, ω^{ij}
- Nondegeneracy $\Rightarrow \omega^{ij}$ has nonzero determinant
- Skew symmetry $\Rightarrow \omega^{ij} = -\omega^{ji}$
- ω closed $\Rightarrow \frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{jk}}{\partial x^i} + \frac{\partial \omega_{ki}}{\partial x^j} = 0.$

Relation between $d\mathcal{H}$, ω, and the hamiltonian vector field

- want a coordinate independent way
- will be related back to the definition presented earlier

Definition

The *Hamiltonian vector field*, denoted $X_{\mathcal{H}}$, is defined by the condition $\iota_{X_{\mathcal{H}}}\omega = d\mathcal{H}$.

- consider the mapping of vector fields to one-forms:

$$X \to \iota_X \omega = \omega(X, \cdot)$$

- gives existence of Hamiltonian vector field

Why Symplectic Geometry?

assuming ω is a 2-form on M such that $\iota_{X_{\mathcal{H}}}\omega = d\mathcal{H}$

- $\iota(X_{\mathcal{H}})\omega = d\mathcal{H}$ should be solvable, so ω should be nondegenerate.
- The flow on M generated by the Hamiltonian vector field should leave ω invariant. Therefore $\mathcal{L}_{X_{\mathcal{H}}}\omega = 0$, so

$$\begin{aligned}\mathcal{L}_{X_{\mathcal{H}}}\omega &= d\iota_{X_{\mathcal{H}}}\omega + \iota_{X_{\mathcal{H}}}d\omega \\ &= dd\mathcal{H} + \iota_{X_{\mathcal{H}}}d\omega = \iota_{X_{\mathcal{H}}}d\omega.\end{aligned}$$

Since ω is nondegenerate by the previous item, ω will be preserved if and only if ω is closed.

- $\mathcal{H}$ should be invariant under the flow of $\mathcal{X}_H$. The last claim can be checked:

$$\mathcal{L}_{X_{\mathcal{H}}}(\mathcal{H}) = \mathcal{X}_{\mathcal{H}}(\mathcal{H}) = d\mathcal{H}(\mathcal{X}_{\mathcal{H}}) = \omega(\mathcal{X}_{\mathcal{H}}, \mathcal{X}_{\mathcal{H}}),$$

which will be zero since ω is a 2-form, so skew-symmetric. The skew symmetry of ω then implies conservation of energy.

Definition

Suppose f and g are two functions on a symplectic manifold (M, ω) which have Hamiltonian vector fields $\mathcal{X}_f$ and $\mathcal{X}_g$ respectively. The *Poisson bracket* of f and g is defined in a coordinate independent manner by

$$\{f, g\} := \omega(\mathcal{X}_f, \mathcal{X}_g) \in C^\infty(M).$$

- The bracket gives smooth functions on the manifold the structure of a Lie algebra.
- $$\{f, g\} = \omega(\mathcal{X}_f, \mathcal{X}_g) = \iota_{\mathcal{X}_g} \iota_{\mathcal{X}_f} \omega = \iota_{\mathcal{X}_g} df = \mathcal{L}_{\mathcal{X}_g} f,$$

  so a function f is constant along integral curves of X_f by the antisymmetry of the Poisson bracket.

Polarization

- What if M is only a symplectic manifold?
- How do we determine which coordinates should play the role of position and which should play the role of momentum?
- *polarization*
- The manifolds we will be concerned with will have additional structure.
- when we need to consider sections of line bundles, the holomorphic and anti-holomorphic sections will provide this distinction.

Geometric Quantization

- constructs quantum analogues of classical systems
- goal is to map observables in a classical setting to observables in a quantum setting
- In quantum mechanics, observables are given by self-adjoint linear operators acting on a Hilbert space.
- not unique

Quantum Mechanics

Definition

A *wave function* in one space dimension is a complex function of position, ψ, such that $|\psi|^2$ is a probability distribution. The probability of finding a given particle between positions a and b is given by $\int_a^b |\psi(q)|^2 \, dq$.

Quantization Requirements

quantization will be denoted by a hat, ˆ
Dirac determined that the map should be linear and have the following properties:

- The identity element of the algebra should map to the identity operator in the Hilbert space.
- Complex conjugation in the Poisson algebra should commute with the mapping.
  star denotes the conjugation of a function and the adjoint of an operator, so $\widehat{(F^*)} = (\hat{F})^*$
  real classical observables will map to Hermitian operators
  eigenvalues of a quantum mechanical observable are the possible measurements, and the measurements should be real, as in the case of a Hermitian operator

Quantization Requirements Continued

- $\{F, G\}$ should map to $[\hat{F}, \hat{G}]_h := \frac{i}{\hbar}(\hat{F}\hat{G} - \hat{G}\hat{F})$.

Definition

A *complete set* is defined as a set $F_1, F_2, ..., F_k$ such that $\{f, F_j\} = 0 \ \ \forall j$ implies f is a constant or $[f, F_j]_h = 0 \ \ \forall j$ implies f is a constant.

- Complete sets should map to complete sets.

Quantization and pre-quantization

- It is a result of Groenwald and van Hove that the quantization conditions cannot all be satisfied for a general symplectic manifold, though quantizations do exist for many specific manifolds
- A mapping which satisfies the four properties above is called a *quantization*.
- If only the first three conditions are satisfied, the map is a *pre-quantization*.

A pre-quantization which is not a quantization

Suppose $M = \mathbb{R}^2 = T^*\mathbb{R}$. We denote the operator of multiplication by x by M_x. Consider the mapping

$$p \to \hat{p} = -\frac{h}{2\pi i}\frac{\partial}{\partial q} \text{ and}$$
$$q \to \hat{q} = M_q + \frac{h}{2\pi i}\frac{\partial}{\partial p}.$$

the fourth condition is not satisfied:
$\{f, q\} = \{f, p\} = 0 \Rightarrow \frac{\partial f}{\partial p} = \frac{\partial f}{\partial q} = 0 \Rightarrow f$ is constant, therefore $\{q, p\}$ form a complete set.
The set of operators $\{\hat{q}, \hat{p}\}$ do not form a complete set - consider either the operator $\frac{\partial}{\partial p}$ or $\frac{\partial}{\partial q} + \frac{2\pi i}{h}p$. Each commutes with $\hat{q}$ and $\hat{p}$, but is not constant.

A quantization of $\mathbb{R}^2$

- consider $\mathcal{H} = L^2(\mathbb{R}^2)$; the operators will be

$$\hat{q} = M_q$$
$$\hat{p} = i\hbar \frac{d}{dq}.$$

- $\hat{q}$ is called the position operator and $\hat{p}$ the momentum operator

The Choice of $\hat{q}$ and $\hat{p}$

- Recall - In the classical case, possible measurements are given by functions of the q^i and p_i variables
- Measurements of quantum mechanical systems are given by eigenvalues of operators.
- We would like to define an operator corresponding to position, and so we will define it such that the position, q^i, is the eigenvalue of the operator $\hat{q}^i$:

$$\hat{q}^i \psi = q^i \psi.$$

- $\hat{p}$ should be the differentiation operator, though our definition of the bracket will require an additional coefficient:

$$\hat{p}_i = i\hbar \frac{\partial}{\partial q^i},$$

Segal's Result, 1960

a quantization of an exact symplectic manifold can be given in the following way:

Theorem (Segal)

Suppose (M, ω) is an exact symplectic manifold with $\omega = d\theta$ and let the vector field $\mathcal{X}_f$ be defined by $\iota_{\mathcal{X}_f}\omega = df$. A quantization is given by

$$\hat{f} = M_f - i\hbar\mathcal{X}_f - \langle \mathcal{X}_f, \theta \rangle$$

An Example Which Is Not A Cotangent Bundle

- consider a phase space given by the sphere, $x_1^2 + x_2^2 + x_3^2 = r^2$ and ω the usual area form, local coordinates are given by $u = x_1, v = x_2$ and the form by $\omega = \dfrac{r(du \wedge dv)}{\sqrt{r^2 - u^2 - v^2}}$. This manifold cannot be represented as a cotangent bundle.
- There exist complex polarizations for this manifold, which comes from a condition on the surface area of the manifold.

- Segal's result allows us to construct a quantization of an exact symplectic manifold
- we want to quantize arbitrary compact manifolds
- We know that the form is always exact locally, so we can cover M by open sets U_α such that on each open set there is a 1-form θ_α where $d\theta_\alpha = \omega$. We could then use Segal's formula on each set.
- IF the manifold admits a quantization, then these local operators can be glued together to form a global operator.

Line Bundles

- We might hope to work with complex holomorphic functions on a phase space
- The only holomorphic functions on compact connected Kähler manifolds are just constants
- Not all manifolds require that we use line bundles - consider $\mathbb{R}^2$

Definition

A *line bundle*, L over a manifold M is a smooth manifold equipped with a smooth surjection $\pi : L \to M$ such that:

- the fibre $\pi^{-1}(m) = L_m \cong \mathbb{C}$ for all $m \in M$, and
- (local triviality) for every $m \in M$, there exists an open neighborhood U_m and a diffeomorphism $\varphi : \pi^{-1}(U_m) \to U_m \times \mathbb{C}$ so that $\phi(L_m) \subset \{m\} \times \mathbb{C}$ and $\varphi|_{L_m}$ is a linear isomorphism.

The simplest example of a line bundle over a manifold, M is $M \times \mathbb{C}$. The vector space at each point m is $\{m\} \times \mathbb{C} \cong \mathbb{C}$.

Definition

A line bundle is called a *trivial line bundle* if there exists a global diffeomorphism $\phi : L \to M \times \mathbb{C}$.

Definition

A *section* of a line bundle is a map $s : M \to L$ such that $\pi \circ s = id_M$.

Proposition

A line bundle is trivial if and only if it has a nowhere vanishing section.

Transition Functions on Line Bundles

- Consider a line bundle L over M
- By local triviality, cover M by open sets $\{U_\alpha\}$ so that each U_α has a nonvanishing section $s_\alpha : U_\alpha \to L$
- this makes $s_\alpha(U_\alpha)$ a trivial line bundle by the proposition
- consider a global section S (possibly vanishing) - one can restrict the global section S to the open sets $\{U_\alpha\}$
- There exist functions $f_\alpha : U_\alpha \to \mathbb{C}$ such that

$$f_\alpha s_\alpha = S|_{U_\alpha}$$

- consider the overlaps between the charts - suppose $U_\alpha \cap U_\beta \neq 0$

$$S|_{U_\alpha \cap U_\beta} = f_\alpha s_\alpha|_{U_\alpha \cap U_\beta} = f_\beta s_\beta|_{U_\alpha \cap U_\beta}$$

- if given a set of nonvanishing sections $\{s_\alpha\}$ and if there exist $\{f_\alpha : U_\alpha \to \mathbb{C}\}$ satisfying $f_\alpha s_\alpha = f_\beta s_\beta$ for all α and β where $U_\alpha \cap U_\beta \neq 0$, then the set $\{s_\alpha\}$ form a global section, S

Definition

The *transition functions* for a line bundle L over M with local sections $\{s_\alpha\}$ over open sets $\{U_\alpha\}$ are functions

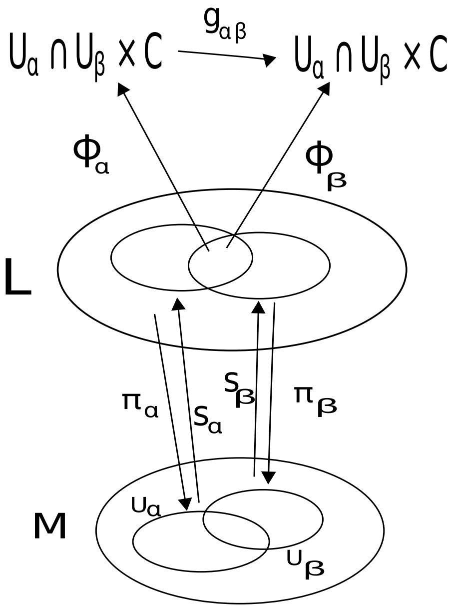
$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C},$$

where, for $z \in \mathbb{C}$, $g_{\alpha\beta}$ is defined by

$$\phi_\alpha \phi_\beta^{-1}(m, z) \to (m, g_{\alpha\beta}(z)).$$

This takes z in the β chart to $g_{\alpha\beta}(z)$ in the α chart, so for sections s_α and s_β,

$$s_\alpha = g_{\alpha\beta} s_\beta.$$



Uα ∩ Uβ x C
gαβ
Uα ∩ Uβ x C
Φα
Φβ
L
πα
Sα
Sβ
πβ
M
Uα
Uβ

Proposition

These transition functions must satisfy the following conditions:

- $g_{\alpha\alpha} = 1$
- $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ *on* $U_\alpha \cap U_\beta$
- $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$ *on* $U_\alpha \cap U_\beta \cap U_\gamma$

These three properties will provide reflexivity, symmetry, and transitivity for an equivalence relation on the disjoint union $\mathbb{C} \sqcup U_\alpha$. The third property is called the *cocycle condition*.

Proposition

Given a manifold M, an open cover $\{U_\alpha\}$, and functions $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}\}$ satisfying the three conditions of the proposition for every U_α and U_β, there exists a line bundle $L \to M$ having $\{g_{\alpha\beta}\}$ as transition functions.

Definition

Line bundles have a *tensor product structure* defined fibre-wise. Suppose L_1 and L_2 are line bundles over M. $L_1 \otimes L_2$ denotes the tensor product of the two line bundles. Elements of this tensor product are equivalence classes such that $[a, b] = [\hat{a}, \hat{b}]$ if $\exists c \in \mathbb{C}$ such that $[\frac{1}{c}a, cb] = [\hat{a}, \hat{b}]$.

- the tensor product of complex line bundles is again a complex line bundle.
- sections are multiplied

Connections

- One motivation for the concept of a connection comes from attempting to differentiate sections of line bundles.
- The connection should measure how a section is changing in the direction of a specific vector field, X.

Definition

A *connection* is a $\mathbb{K}$-linear (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) map $\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$ such that $\forall f \in C^\infty(M), u \in C^\infty(E), \nabla(fu) = df \otimes u + f\nabla u$.

- The connection along a vector field X is a map $\nabla_X : \Gamma(L) \to \Gamma(L)$.
- We will think of connections as assignments of 1-forms.

The definition of the connection implies the following properties:

- The map to respect scaling by functions on the manifold, so we will require

$$\nabla_{fX} u = f \nabla_X u \quad \forall f \in C^\infty(M).$$

- The connection ∇ should satisfy a Leibniz (product) rule,

$$\nabla_X (fu) = (Xf)u + f \nabla_X u \forall f \in C^\infty(M), u \in C^\infty(E).$$

- The connection should satisfy

$$\nabla_{X+Y} \phi = \nabla_X \phi + \nabla_Y \phi$$

If the manifold M is also equipped with a metric, h, then it is natural to require that the two structures interact in a specified way.

Definition

A connection is said to be *compatible with the metric h* if

$$\nabla_X(h(Y,Z)) = h(\nabla_X Y, Z) + h(Y, \nabla_X Z)$$

A natural Hermitian metric to keep in mind is on the trivial bundle:

$$h(m, z_1), (m, z_2) = z_1\bar{z}_2 \ \forall (m, z_i) \in (M, \mathbb{C}).$$

Proposition

Every line bundle admits a connection.

Definition

Suppose ∇ is a connection on a line bundle L over M. Consider a local trivialization $\phi_i : U_i \to \mathbb{C}$. The connection acts on open sets by determining a 1-form called the *potential 1-form*. On a given open set U with connection ∇, the connection acts on a section s by the rule

$$\nabla s = -i\theta s.$$

Because it will arise often, we denote

$$\eta = -i\theta.$$

Proposition

The connection ∇ can be globally defined.

Curvature

Definition

The *curvature 2-form* of the connection ∇ with potential 1-form θ is given by $d\theta$. The curvature $d\theta$ will be denoted Ω.

Proposition

The curvature, Ω, is globally defined.

Idea: Suppose that on U_0, $\nabla(s_0) = \eta s_0$ and on U_1, $\nabla(s_1) = \hat{\eta} s_1$. The transition between the sections is given by $s_1 = \phi s_0$ on $U_0 \cap U_1$.

$$(\hat{\eta} - \eta)(X) = \frac{d\phi(X)}{\phi} = d\ln(\phi)(X)$$

In other words, changing the chart corresponds to mapping the potential 1-forms by the rule

$$\eta \mapsto \eta + d\ln\phi.$$

We have a complex line bundle L with Hermitian structure h and compatible connection ∇. We can choose the trivialization to satisfy $h(s_j, s_j) = 1 \ \forall j$ by simply rescaling.

Proposition

The one-form η is purely imaginary.

Cech Cohomology

We need to relate the global structure of quantization with the local structure that Segal's result gives. We will construct this relation with Cohomology.

Definition

An open cover $\{U_\alpha\}$ of M is called a *contractible cover* of M if each of the open sets $U_i, U_i \cap U_j, U_i \cap U_j \cap U_k \ldots$ is either empty or can be smoothly contracted to a point.

Definition

A *k-simplex* is a $k+1$-tuple of indices $(i_0, i_1, \ldots, i_k)$ determining sets $(U_{i_0}, U_{i_2}, \ldots, U_{i_k})$ so that $U_{i_0} \cap U_{i_2} \cap \ldots \cap U_{i_k} \neq 0$.

Definition

A *k-cochain* is a totally skew (skew in any pair of coordinates) map

$$g : (i_0, i_1, \ldots, i_k) \mapsto g(i_0, i_1, \ldots, i_k) \in \mathbb{R}.$$

The set of all k-cochains will be denoted $C^k(U, \mathbb{R})$.

- The target space of a k-cochain is generally defined to be an abelian Lie group.
- For the purposes of this section, we will only need this group to be $\mathbb{R}$.
- We will also only need the case $k = 2$ for our construction.

Definition

For each k, there is a map $\delta : C^{k+1}(U, \mathbb{R}) \to C^k(U, \mathbb{R})$ defined, using $\hat{}$ to denote omission, by

$$\delta g(i_0, i_1, \dots, i_{k+1}) = \sum_{j=0}^{k+1} (-1)^j g(i_0, i_1, \dots, \hat{i}_j, \dots, i_{k+1})$$

The operator δ is called the *coboundary operator*.

Definition

If $g \in C^k(U, \mathbb{R})$ and $\delta g = 0$, then g is called a *k-cocycle*.

Definition

If g is a k-cocycle, and $g = \delta h$ for some $(k+1)$-cochain, h, then g is called a *k-coboundary*. The set of coboundaries is denoted $Z^k(U, \mathbb{R})$.

Definition

The k^{th} *cohomology group* relative to $\{U\}$ is defined as the quotient

$$H^k(U, \mathbb{R}) = C^k(U, \mathbb{R})/\delta(C^{k-1}(U, \mathbb{R})).$$

The elements of this group consist of equivalence classes of cocycles with the relation that two cocycles are in the same class if they differ by a coboundary.

- The Cech cohomology of a manifold is independent of the choice of contractible cover. Because of this, we may denote the cohomology by $H^k(M, \mathbb{R})$.
- The cohomology groups discussed here are isomorphic with the de Rham cohomology groups.

The Integrality Condition

A form ω is called *integral* if the class of $(2\pi\hbar)^{-1}\omega$ lies in the image of $H^2(M, \mathbb{Z})$.

Definition

A symplectic manifold (M, ω) is called *quantizable* if ω satisfies the integrality condition above, i.e. ω is integral.

This is equivalent to:

Proposition

A symplectic manifold is quantizable if there exists a Hermitian line bundle $B \to M$ with a connection ∇ on B which has curvature $\hbar^{-1}\omega$.

The first definition requires the integral of ω over any closed 2-dimensional surface in M be integer multiple of $2\pi\hbar$. The B in the second definition is called the *quantizing line bundle*.

Quantization of Kähler Manifolds

Definition

A *complex manifold*, M, is a smooth manifold which has coordinate patches diffeomorphic with $\mathbb{C}^n$ for some fixed n which has holomorphic transition functions.

Definition

A *Hermitian metric* is an assignment of a self-adjoint (Hermitian) form to each fibre of a line bundle.

Definition

Suppose M is a complex manifold which has a smoothly varying Hermitian metric h on its tangent spaces. If the imaginary part of h is closed, M is called a *Kähler manifold*.

- The Hermitian metric relates structures on the manifold.
- imaginary part of h is a non-degenerate, skew-symmetric 2-form. If 2-form is also closed $\Rightarrow$ it is a symplectic form
- real part of h assigns a symmetric 2 form to each of the tangent spaces $\Rightarrow$ Riemannian manifold.

Definition

Notation

$$\partial = \sum_i dz_i \frac{\partial}{\partial z_i}$$

$$\bar{\partial} = \sum_i d\bar{z}_i \frac{\partial}{\partial \bar{z}_i}$$

This notation allows the immediate decomposition of d as $\partial + \bar{\partial}$. By considering only holomorphic sections, we now have the condition $\text{curv}(L) = -\partial\bar{\partial}\ \ln(h_0) = \bar{\partial}\partial\ \ln(h_0)$.

$\mathbb{C}^n$

If $\mathbb{C}^n$ is endowed with the natural symplectic form $\omega = i\sum_j dz_j \wedge d\bar{z}_j$, the line bundle is trivial and we can find the appropriate Hilbert spaces for the quantization. Since $i \cdot \mathrm{curv}(L) = \omega$, $\omega = -i\partial\bar{\partial}\ln(h_0)$. Therefore,

$$\partial\bar{\partial}\ln(h_0) = -\omega = -\sum_j dz \wedge d\bar{z}.$$

However, $\partial\bar{\partial}z\bar{z} = \sum_j dz \wedge d\bar{z}$, so

$$\ln(h_0(z)) = -z\bar{z}.$$

Therefore,

$$h_0(z) = e^{-z\bar{z}} = e^{-|z|^2}.$$

- arbitrary powers of a line bundle over a manifold gives $h(z) = e^{-k|z|^2}$ for any positive integer k.
- transition functions of a tensor product of line bundles are products of the original transition functions, so the transition functions of $L^{\otimes n}$ are just the powers of the transition functions of the line bundle L.
- The Hermitian structure is then given locally by h_0^k. The Hilbert spaces are then

$$\mathcal{H}_k = L^2_{hol}(\mathbb{C}, e^{-k|z|^2} dz \wedge d\bar{z}).$$

- The curvature of the k^{th} tensor power of L is given by

$$\text{curv}(L^{\otimes k}) = k\,\text{curv}(L).$$

- This gives us sets of Hilbert spaces defined by

$$\mathcal{H}_k := L^2_{hol}(M, L^{\otimes k}),$$

Quantization of $\mathbb{S}^2$

We will consider the Riemann sphere with two charts given by stereographic projection.

Proposition

The metric on a section s is given by

$$h(s(z), s(z)) = \int s(z)s(\bar{z})\frac{dz \wedge d\bar{z}}{(1+|z|^2)^{2+n}}.$$

special case of the Fubini-Study fundamental form for projective space, defined as

$$\omega_{FS} = i\frac{(1+|w|^2)\sum_{i=1}^{N} dw_i \wedge d\bar{w}_i - \sum_{i,j=1}^{N} \bar{w}_i w_j dw_i \wedge d\bar{w}_j}{(1+|w|^2)^2}.$$

An Orthonormal Basis

Proposition

The dimension of the space of holomorphic sections of $L^{\otimes n}$ is n.

Proposition

A basis for the Hilbert space of holomorphic sections of $L^{\otimes n}$ is given by

$$s_k = \sqrt{\frac{(n+1)}{2\pi}\binom{n}{k}}z^k$$

for $k = 1, \ldots, n$.

To summarize, the Hilbert space for the quantization of $\mathbb{S}^2$ is equivalent to the space

$$\mathcal{H}_N^S = \left\{ f : \mathbb{C} \to \mathbb{C} \mid f \text{ is entire and } \frac{i}{2} \int_{\mathbb{C}} |f(z)|^2 \frac{dz \wedge d\bar{z}}{(1+|z|^2)^{N+2}} < \infty \right\}.$$

This identification is achieved through stereographic projection (by considering the sphere to be the completion of $\mathbb{C}$). These functions are just homogeneous polynomials with degree less than or equal to N.

Quantization of $\mathbb{T}^2$

The functions on $\mathbb{T}^2$ are equivalent to doubly periodic functions on the plane. The area form on the torus will then take a much simpler form than the area form on $\mathbb{S}^2$ did, namely

$$\omega_N = Nh\, dx \wedge dy \mod \mathbb{Z} \times \mathbb{Z}$$

To satisfy the quantization condition, we need a 1-form θ such that $d\theta = h^{-1}\omega$, so we will use

$$\phi^*(\theta) = Nhx\, dy$$

to achieve this given our maps. We will relate the sections with another set of functions, the *theta functions*.

Consider the set of functions $f(z)$ satisfying the condition

$$f(z+m+in) = e^{N\pi(n^2-2inz)} f(z)$$

for a fixed value of N.

Definition

The set of functions that satisfy the transition function condition will be denoted Θ_N. These will be called the theta functions of order N.

Proposition

The set of functions Θ_N has dimension N.

Proposition

The functions θ_j, defined by the Fourier series

$$\theta_j = \sum_{k=-\infty}^{\infty} e^{-\pi(Nk^2+2jk)} e^{2\pi izj+2\pi iNk}$$

for $j = 0, \ldots, N-1$ define an orthogonal basis for Θ_N with inner product

$$h(f,g) = \int_{[0,1]\times[0,1]} f(z)\bar{g}(z)e^{-2N\pi y^2}dxdy.$$

Significant in this computation is the fact that

$$||\theta_j||^2 = \frac{e^{2\pi j^2/N}}{\sqrt{2N}}.$$

This allows us to construct an orthonormal basis.

Toeplitz Operators

- Toeplitz operators will be the tools used to estimate solutions to the Toda lattice.
- M is a Kähler manifold with symplectic form ω.
- M must be quantizable, so $\frac{\omega}{2\pi\hbar}$ must be integral.
- L is a holomorphic Hermitian line bundle over M with connection ∇ with curvature form equal to ω.

Definition

Define the Hilbert spaces

$$\mathcal{H}_N = H^0(M, L^{\otimes N})$$

for each N. These are the spaces of holomorphic sections of the N^{th} tensor power of L.

Definition

We will denote orthogonal projection by

$$\Pi : L^2(M, L^{\otimes N}) \to \mathcal{H}_N.$$

So that we can discuss the case of the torus and sphere simultaneously, we will denote the Hilbert space on the sphere by $\mathcal{H}_N^S$ and the Hilbert space on the torus by $\mathcal{H}_N^T$.

- sphere - the Hilbert space $\mathcal{H}_N^S$ is a closed subspace of the space $L^2(\mathbb{C}, \frac{i}{2}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^{N+2}})$
- torus - the Hilbert space $\mathcal{H}_N^T$ is a closed subspace of $L^2([0,1]^2, e^{-2N\pi y^2} dx \wedge dy)$
- Having defined the basis for $\mathcal{H}_N^S$, the projection is naturally defined. For $\mathbb{T}^2$, $\mathcal{H}_N^T$ is a closed subspace of $L^2([0,1] \times [0,1])$, and so there is again a natural projection.

Definition

A *Toeplitz operator* T is a sequence of operators $T_H^{(N)}$ acting on $\mathcal{H}_N$ which has an asymptotic expansion of the form

$$T_H^{(N)} \sim \sum_{j=0}^{\infty} N^{-j} T_{H_j}^{(N)},$$

where H_j are functions such that

$$\begin{aligned} T_{H_j}^{(N)} : \mathcal{H}_N &\to \mathcal{H}_N \\ f &\to \Pi(fH_j). \end{aligned}$$

Example

The space on which we will be working ($\mathcal{H}_N$ above) will be the span of $1, e^{ix}, e^{2ix}, \dots, e^{Nix}$. We would like to understand $T_H(f)$, which we will find by considering the series $T_H^{(N)}(f)$. First, we can project f onto the span of $1, e^{ix}, e^{2ix}, \dots, e^{Nix}$, producing

$$\Pi(f) = \langle f, 1\rangle \cdot 1 + \langle f, e^{ix}\rangle \cdot e^{ix} + \dots + \langle f, e^{Nix}\rangle \cdot e^{Nix}.$$

To keep track of these coefficients, we set $c_0 = \langle f, 1\rangle$, $c_1 = \langle f, e^{ix}\rangle, \dots, c_N = \langle f, e^{Nix}\rangle$. We now need to multiply $\Pi(f)$ by $H = \sum_{m=-\infty}^{\infty} k_m e^{imx}$.

The result is then projected back to the span of $1, e^{ix}, e^{2ix}, \ldots, e^{Nix}$. We can represent this entire process by considering a matrix. The matrix of $T_H^{(N)}$ will be an $N \times N$ block of a doubly infinite matrix which is operating on the Fourier series of the function f. In our example, this matrix has the form

$$\begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\
\cdots & 0 & k_0 & k_{-1} & k_{-2} & \dots & k_{-N+1} & 0 & \cdots \\
\cdots & 0 & k_1 & k_0 & k_{-1} & \dots & k_{-N+2} & 0 & \cdots \\
\cdots & 0 & k_2 & k_1 & k_0 & \dots & k_{-N+3} & 0 & \cdots \\
\cdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & 0 & k_{N-1} & k_{N-2} & \cdots & k_1 & k_0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & &
\end{pmatrix}.$$

The Toeplitz operator is then defined by this sequence.

- Toeplitz operators can arise when we try to do operations on holomorphic sections of a line bundle.
- If a holomorphic section is multiplied by a differentiable function, the resulting section is only differentiable, so one needs to project back to the holomorphic sections.
- Because of our definition in terms of having an asymptotic expansion, and since $\mathcal{H}_N$ is a finite dimensional space, being a Toeplitz operator is only seen in the limit $N \to \infty$.

Definition

The *order of a Toeplitz operator* is the first $j \geq 0$ for which $H_j \neq 0$.

- The composition of two Toeplitz operators is a Toeplitz operator with order equal to the sum of the orders of the summands.
- The commutator is also a Toeplitz operator with order one less than the sum of orders of the summands.

Proposition

If $T = (T^{(N)})$ *is a Toeplitz operator of order m, then*

$$||T^{(N)}||_{HS} \sim d_N N^{-m} ||H_m||_2,$$

where $||T^{(N)}||_{HS}$ *is the Hilbert-Schmidt norm and* d_N *is the dimension of* $\mathcal{H}_N$.

Definition

If a Toeplitz operator T has the expansion $\sum_{j=0}^{\infty} \Pi_N M_{H_j} \Pi_N$, then H_0 is called the *principal symbol* of T.

Toeplitz Quantization of $\mathbb{T}^2$

Lemma

The principal symbol of the Toeplitz operator associated to the matrix L_N for the period Toda lattice is given by

$$H(x, y) = b(x) + 2a(x)\cos(2\pi y),$$

where x and $2\pi y$ are the natural coordinates on the torus.

- compute the Fourier coefficients of a function of the form

$$H(x, y) = u(y) + 2v(y)\cos(2\pi x)$$

  in the orthonormal bases defined above.
- The u and b function will eventually be equated and the v and a functions will be closely related.

We will break the function into its two summands and consider each separately in the following two propositions, which will then be used to prove the lemma.

Proposition

If $v(y) = 0$, so that $H(x, y) = u(y)$, the Toeplitz matrix for H is diagonal with the j^{th} diagonal entry given by

$$\lambda_j^{(N)} = \sum_{m=-\infty}^{\infty} \hat{u}(m) e^{-\pi m^2/2N} e^{-2\pi i m j/N}.$$

As $N \to \infty$, if $\hat{u}$ has compact support in m,

$$\lambda_j^{(N)} \sim \sum_m \hat{u}(m) e^{-2\pi i m j/N} = u(-j/N)$$

uniformly in j.

Proposition

If $H(x,y)$ is of the form $2\cos(2\pi x)v(y)$, then the Toeplitz matrix for H is 0 except for the super and sub diagonals and corners. The entries of the matrix are asymptotic to the entries of the matrix

$$\begin{pmatrix} 0 & v(1-1/2N) & 0 & \cdots & v(1/2N) \\ v(1-1/2N) & 0 & v(1-3/2N) & \cdots & 0 \\ 0 & v(1-3/2N) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & v(3/2N) \\ v(1/2N) & 0 & 0 & v(3/2N) & 0 \end{pmatrix}.$$

we obtain the matrix

$$\begin{pmatrix} u(1) & v(1-1/2N) & 0 & \cdots & v(1/2N) \\ v(1-1/2N) & u(1-1/N) & v(1-3/2N) & \cdots & 0 \\ 0 & v(1-3/2N) & u(1-2/N) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & v(3/2N) \\ v(1/2N) & 0 & 0 & v(3/2N) & u(1/N) \end{pmatrix}.$$

We now take

$$u(x) = b(x)$$

and

$$v(x) = a\left(x - \frac{1}{2N}\right).$$

using the basis $\{(2N)^{1/4}e^{-\pi(N-1)^2/N}\theta_{N-1}, \ldots, (2N)^{1/4}\theta_0\}$, which is just a rearrangement of the original basis. Now with $a_j = a(\frac{j}{N})$ and $b_j = b(\frac{j}{N})$, we obtain the desired matrix

We can also see that the operator we have constructed is, in fact, a Toeplitz operator.

- we have functions u and v.
- We can find functions u_N and v_N so that $u_N \sim u + \sum_{j=1}^{\infty} N^{-j} u_j$ and $v_N \sim v + \sum_{j=1}^{\infty} N^{-j} v_j$.
- We then have our H_N functions which will be given by $H_N(x, y) = u_N(y) + 2\cos(2\pi x) v_N(y)$. Now consider the matrix

$$\mathcal{T}_k^{(N)} = \begin{pmatrix} b_k(1/N) & a_k(1/N) & 0 & \dots & 0 \\ a_k(1/N) & b_k(2/N) & a_k(2/N) & \dots & 0 \\ 0 & a_k(2/N) & b_k(3/N) & a_k(3/N)\dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0\cdots & \cdots & a_k(1-1/N) & b_k(1) & \end{pmatrix}$$

For the function H, we have now constructed an operator $T_{H,K}$ which has an asymptotic expansion

$$T_{H,K} = \sum_{k=0}^{K} N^{-k} \mathcal{T}_k^{(N)} + O(N^{-k+1}).$$

These are truncations of the operator T_N.

Lemma

The matrices L_N above are the $T^{(N)}$ in the definition of a Toeplitz operator. Written as a function of h and θ, the height and angle of the sphere, the principal symbol is given by

$$H_0(h,\theta) = b(h) + 2a(h)cos(\theta).$$

Dispersion

To understand dispersion, we will investigate its appearance in a particular setting. The following equation was studied by Korteweg and de Vries in 1895 to describe shallow water waves:

Definition (The KdV Equation)

$$u_t = uu_x + u_{xxx}$$

- $u_t = \frac{\partial u}{\partial t}$, etc.
- In a physical situation, we will have nontrivial coefficients
- The terms on the right hand side of the equation correspond to different aspects of the wave, depending on the relation of the height of a wave to the depth of the water.

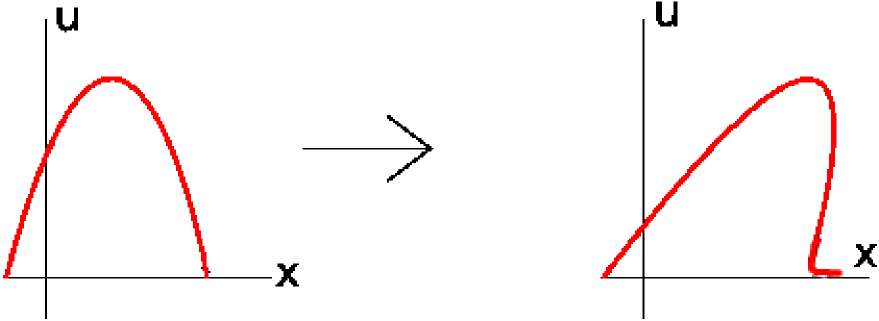
- If the water depth is much larger than the amplitude of the wave, we can approximate the equation by the *linear KdV equation*:

$$u_t = u_{xxx}$$

- For waves where the amplitude is much larger than the depth of the water, the term u_{xxx} becomes insignificant, so the wave can be modelled by the equation

$$u_t = uu_x$$

The height of the wave is given by u, so u_t is the speed of the wave. The speed of the wave will depend on the height of the wave, so points with larger height with move faster. This will force a wave to topple.



Solving the nonlinear equation

We can check that if u is implicitly given by $u = f(x + ut)$, then this provides a solution. An inspiration for this solution is to instead consider a solution to $u_t = cu_x$. This guess is valid as long as $\frac{\partial f}{\partial u} \neq 0$, precisely the moment when we have a vertical tangent line to the function f, as the wave is about to topple.

Solving the linear KdV equation

We will next consider the situation given by the linear KdV equation. This equation can be solved by methods of Fourier analysis, by assuming a solution u and considering its Fourier transform.

We expect the solution to be of the form

$$\int \hat{f}(k) e^{i(kx+\omega t)} dk,$$

where $kw + \omega t$ is the phase function.

- k is the *wave number*, and is the spatial analogue of the wave frequency.
- If use this guess, we see that $\omega = k^3$.
- $\dfrac{\omega}{k}$ is the *propagation speed* or *phase speed* and describes the speed of the phase function.
- This is seen by considering the initial phase function, $\theta_0 = kx + \omega t$. The position, x, is then given by $x = \dfrac{\theta_0 - \omega t}{k}$. $\frac{\omega}{k}$ is then the speed.

Definition

A solution which has a propagation speed dependent on the wave number is said to be *dispersive*.

In our case, the propagation speed is given by k^2. $u_t = uu_x$ does not have this property, so it is also referred to as the *dispersionless KdV equation*.

We will also mention one of the important solutions to the KdV equation, the soliton. This is given by

$$u(x,t) = \frac{1}{2}c^2 \operatorname{sech}^2(\frac{1}{2}c(x + c^2 t)),$$

and this is particularly remarkable since the presence of a solitary wave solution can only be achieved by a balancing of the dispersive and nonlinear summands of the KdV equation.

Statements of the Theorems

Setup:
Let $a, b \in C^\infty(\mathbb{R})$ be 1-periodic functions, and let a_j^t and b_j^t, $j = 1, \ldots, N$ be the solution of the periodic Toda flow

$$\dot{a}_j = a_j(b_{j+1} - b_j)$$
$$\dot{b}_j = 2(a_j^2 - a_{j-1}^2)$$

with initial conditions

$$a_j^{t=0} = a\left(\frac{j}{N}\right)$$
$$a_j^{t=0} = b\left(\frac{j}{N}\right).$$

Let us moreover suppose that there exists $s_c > 0$ such that the system

$$\partial_s a = a\partial_x b$$
$$\partial_s b = 2\partial_x a^2$$

with initial conditions

$$a^{s=0}(x) = a(x)$$
$$b^{s=0}(x) = b(x)$$

has a smooth periodic solution for $s < s_c$.

Theorem

There exist two sequences of smooth functions (determined by $a^s(x)$ *and* $b^s(x)$*),* $a_k^s(x)$ *and* $b_k^s(x)$*,* $k = 1, 2, \ldots,$ *defined on* $[0, s_c) \times \mathbb{R}$ *and periodic in* x*, such that for all integers* $K > 0$ *and for each* $\epsilon > 0$*, there exist* $C_k > 0$ *such that for* $t \leq N(s_c - \epsilon)$*,*

$$\forall j = 1, \ldots, N, \quad \left| a_j^t - \left(a^{\frac{t}{N}}\left(\frac{j}{N}\right) + \sum_{k=1}^{K-1} N^{-k} a_k^{\frac{t}{N}}\left(\frac{j}{N}\right) \right) \right| \leq C_K N^{-K}$$

and

$$\forall j = 1, \ldots, N, \quad \left| b_j^t - \left(b^{\frac{t}{N}}\left(\frac{j}{N}\right) + \sum_{k=1}^{K-1} N^{-k} b_k^{\frac{t}{N}}\left(\frac{j}{N}\right) \right) \right| \leq C_K N^{-K}.$$

Theorem (Theorem continued)

In particular, as $N \to \infty$, $\frac{j}{N} \to x$, *and* $\frac{t}{N} \to s < s_c$,

$$a_j^t \longrightarrow a^s(x)$$

and

$$b_j^t \longrightarrow b^s(x)$$

where a^s *and* b^s *are solutions of the Toda PDEs.*

Second Theorem

Setup: Let

$$a \in \mathcal{A} = \left\{ a \in C^0([0,1]) \;\middle|\; \frac{a(x)}{\sqrt{x(1-x)}} \in C^\infty([0,1]) \right\}$$

and $b \in C^\infty(\mathbb{R})$, and let a_j^t for $j = 1, \ldots, N-1$ and b_j^t for $j = 1, \ldots, N$ be the solution of the non-periodic Toda flow

$$\dot{a}_j = a_j(b_{j+1} - b_j)$$
$$\dot{b}_j = 2(a_j^2 - a_{j-1}^2)$$

with initial conditions

$$a_j^{t=0} = a\left(\frac{j}{N}\right)$$
$$a_j^{t=0} = b\left(\frac{j}{N}\right).$$

Let us moreover suppose that there exists $s_c > 0$ such that the system

$$\partial_s a = a\partial_x b$$
$$\partial_s b = 2\partial_x a^2$$

with initial conditions

$$a^{s=0}(x) = a(x)$$
$$b^{s=0}(x) = b(x)$$

has a solution with $a^s \in \mathcal{A}$ and $b^s \in C^\infty)[0,1])$, for $s < s_c$.

Theorem

There exist two sequences of smooth functions (determined by $a^s(x)$ *and* $b^s(x)$*),* $a^s_k(x)$ *and* $b^s_k(x)$*,* $k = 1, 2, \dots,$ *defined on* $[0, s_c) \times (0, 1)$ *with* $a^s_k(\cdot) \in \mathcal{A}$ *for each* $s \in [0, s_c)$*, such that for all integers* $K > 0$ *and for each* $\epsilon > 0$*, there exist* $C_k > 0$ *such that for* $t \leq N(s_c - \epsilon)$*,*

$$\forall j = 1, \dots, N-1, \quad \left| a^t_j - \left(a^{\frac{t}{N}}\left(\frac{j}{N}\right) + \sum_{k=1}^{K-1} N^{-k} a_k^{\frac{t}{N}}\left(\frac{j}{N}\right) \right) \right| \leq C_K N^{-K}$$

and

$$\forall j = 1, \dots, N, \quad \left| b^t_j - \left(b^{\frac{t}{N}}\left(\frac{j}{N}\right) + \sum_{k=1}^{K-1} N^{-k} b_k^{\frac{t}{N}}\left(\frac{j}{N}\right) \right) \right| \leq C_K N^{-K}.$$

Theorem

Theorem Continued In particular, as $N \to \infty$, $\frac{j}{N} \to x$, *and* $\frac{t}{N} \to s < s_c$,

$$a_j^t \to a^s(x)$$

and

$$b_j^t \to b^s(x)$$

where a^s *and* b^s *are solutions of the Toda PDEs.*

Sketch of the Proofs

Lemma

Let $L(t)$ satisfy

$$\frac{dL}{dt} = [L(t), B(L(t))]$$

where $B(L(t))$ is defined according to the whether the periodic or non-periodic lattice is being considered. We also require $L(0)$ to be given by the appropriate matrix. Let s_c be defined as in the theorem above. For all $s < s_c$, there exists a Toeplitz operator T_s such that

$$||L(Ns) - T_s||_{HS} = O(N^{-\infty})$$

where $|| \cdot ||_{HS}$ is the Hilbert Schmidt norm. Despite the change in notation, this norm is still evaluated on $\mathcal{H}_N^S$ and $\mathcal{H}_N^T$. Furthermore, for all $\epsilon > 0$, the estimates on $[0, s_c - \epsilon]$ are uniform.

- The proof is constructive, and defines the operator in an inductive way.
- We are given $L(t)$ and so must construct T_s.
- To do this, we must find a symbol H so that T_H is "close" to L.
- Suppose L_1 and L_2 are Toeplitz operators defined as in the Toda lattice case. This means that the operators have principal symbols given by

$$H^{1,2}(\theta, h) = a_{1,2}(h) + 2\cos(\theta)a_{1,2}(h).$$

- It is true that $L_1 L_2$ is also a Toeplitz operator with principal symbol $H^1 H^2$ and $\frac{1}{iN}[H^1, H^2]$ is a Toeplitz operator with principal symbol given by

$$\{H_0^1, H_0^2\} = \partial_h H_0^1 \partial_\theta H_0^2 - \partial_\theta H_0^1 \partial_h H_0^2.$$

We now apply this to our operator T_H. If we use x to denote either h or ϕ depending on the situation, then T_H has principal symbol $H(x, \theta) = b(x) + 2a(x)cos(\theta)$ and $\frac{1}{iN}B(T_H)$ has principal symbol $-2\partial_\theta a(x)\cos(\theta)$.

The next step is to construct a certain smooth one-parameter family of self-adjoint Toeplitz operators, denoted $\Lambda(t)$. We will let X denote the sphere or the torus so that the following theorem applies to either space.

Proposition

Let $L_0 = \{L_0^{(N)}\}$ be a self-adjoint Toeplitz operator on X, tridiagonal in the standard basis with principal symbol $H_0 : X \to \mathbb{R}$. Let $J = [0, \tau]$ be a closed one-sided neighborhood of zero in $\mathbb{R}$, and assume that there exists a solution $H : J \times X \to \mathbb{R}$ of the initial value problem

$$\begin{cases} \frac{\partial}{\partial s} H = \{H, \partial_x H\} \\ H|_{s=0} = H_0. \end{cases}$$

Then there exists a smooth one-parameter family of self-adjoint operators, $\Lambda(t)$, of order zero, with $\Lambda^{(N)}(t)$ defined for $\frac{t}{N} \in J$ and Toeplitz operators R and S such that

$$\begin{cases} \frac{d}{dt} \Lambda = [\Lambda, B(\Lambda)] + R \\ \Lambda|_{t=0} = L_0 + S. \end{cases}$$

Proposition

Proposition continued The norms of R and S are of arbitrary order in N^{-1}. Moreover, Λ can be chosen to be tridiagonal.

- make increasingly accurate approximations of Λ
- begin with $\Lambda_0(s)$, which is self-adjoint and time dependent
- choose $\Lambda_0(s)$ to be an order zero Toeplitz operator with principal symbol H, making it the most general approximation we could start with
- $\Lambda_0(s)$ can be chosen to be tri-diagonal
- For a Toeplitz operator $\mathcal{R}_0$ of order -1,

$$\frac{d}{ds}\Lambda_0 = N[\Lambda_0, B(\Lambda_0)] + \mathcal{R}_0.$$

- construct a chain of Λ_i such that the norms of R_i and S_i are of increasing order in N^{-1}. This will allow us to create Λ, R, and S of arbitrary order

- Now, with $\mathcal{S}$ a Toeplitz operator of order -1 in N, define

$$\Lambda_1 = \Lambda_0 + \mathcal{S}.$$

- We now get

$$\frac{d}{ds}\Lambda_1 = N[\Lambda_1, B(\Lambda_1)] + \mathcal{R}_1$$

  where $\mathcal{R}_1$ is a Toepltiz operator of order -2 as long as the symbol σ of S satisfies

$$\frac{d}{ds}\sigma = \{H, \partial_\theta \sigma\} + \{\sigma, \partial_\theta H\} - \rho_0$$

  where ρ_0 is the principal symbol of $\mathcal{R}_0$. This is a linearization of the equation above around H.

- Since $\Lambda_0(s)$ was tri-diagonal, σ will also be tri-diagonal. This makes the equation solvable with smooth solutions (since it is a hyperbolic first-order 2x2 system).

Proceeding by this method, we can construct Λ_∞ so that

$$\begin{cases} \Lambda_\infty|_{s=0} = L_0 + O(N^{-\infty}) \\ \frac{d}{ds}\Lambda_\infty - N[\Lambda, [\Lambda_\infty, \mathcal{Z}]] = O(N^{-\infty}). \end{cases}$$

This is very nearly our desired result. We only need to make the change of variables $t = sN$.

We now want to relate the operator Λ_∞ to the Toda lattice. This will be shown by comparing the matrices and considering an energy norm. We will then denote the matrix of Λ_∞ by

$$\Lambda_\infty = \begin{pmatrix} B_1 & A_1 & 0 & \cdots & 0 & A_N \\ A_1 & B_2 & A_2 & 0 & \cdots & 0 \\ 0 & A_2 & B_3 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_N & \cdots & \cdots & \cdots & A_{N-1} & B_N \end{pmatrix},$$

where A_N is 0 for the non-periodic lattice.

We can then consider the difference,

$$L(t) - \Lambda_\infty(t) = \begin{pmatrix} \beta_1 & \alpha_1 & 0 & \cdots & 0 & \alpha_N \\ \alpha_1 & \beta_2 & \alpha_2 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \beta_3 & \alpha_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_N & 0 & \cdots & 0 & \alpha_{N-1} & \beta_N \end{pmatrix}.$$

we can achieve bounds on the energy function

$$E := \sum_{j=1}^{N} \left(2\alpha_j^2 + \frac{1}{2}\beta_j^2 \right)$$

where α_i and β_i are defined as above. Furthermore, Bloch et al show that $E(t=0) = O(N^{-\infty})$. Bloch et al go on to show by computation that $E = O(N^{-\infty})$ uniformly for $\frac{t}{N}$ bounded. Since E controls the Hilbert-Schmidt norm of $L(t) - \Lambda_\infty$, the lemma is proven.

Proof of the Main Theorems

- we know that there exists a Toeplitz operator T_s such that $||L(Ns) - T_s||_{HS} = O(N^{-\infty})$ uniformly on $[0, s_c - \epsilon]$ for every $\epsilon > 0$
- now use the functions which were created in the matrix $\mathcal{T}_k$ to approximate the true solutions to the Toda lattice
- Recall

$$\mathcal{T}_k = \begin{pmatrix} b_k(1/N) & a_k(1/N) & 0 & \dots & 0 \\ a_k(1/N) & b_k(2/N) & a_k(2/N) & \dots & 0 \\ 0 & a_k(2/N) & b_k(3/N) & a_k(3/N)\dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0\cdots & \cdots & a_k(1-1/N) & b_k(1) & \end{pmatrix}.$$

These matrices were used to construct the Toeplitz operator $T_{H,K}$, and

$$||L(t) - T_{H,N}||_{HS} = O(N^{-(K+1)}).$$

- compare $L(t)$ with $\mathcal{T}_{k-1}$, and in particular the entries in each matrix, since we are considering a Hilbert-Schmidt norm
- bound the difference between exact solutions of the Toda lattice given by a_j^t and the approximation by the Toeplitz operator, giving the relations and limits of the theorem