Consecutive Sums

Matt Thomas

What are consecutive sums?

Consecutive sums are sums of the form $a + (a+1) + (a+2) + \cdots + b$.

What are consecutive sums?

Consecutive sums are sums of the form $a + (a+1) + (a+2) + \cdots + b$.

What patters did you find when investigating consecutive sums?

Odd Numbers

Odd Numbers

Proposition

Any odd number may be written as a sum of two consecutive integers.

Odd Numbers

Proposition

Any odd number may be written as a sum of two consecutive integers.

Proof.

Consider an odd number N. Both $N+1$ and $N-1$ are even, so $\frac{N+1}{2}$ and $\frac{N-1}{2}$ are both integers. $\frac{N+1}{2}$ and $\frac{N-1}{2}$ are consecutive since

$$\frac{N-1}{2} + 1 = \frac{N-1}{2} + \frac{2}{2} = \frac{N+1}{2}. \tag{2}$$

Finally,

$$\frac{N-1}{2} + \frac{N+1}{2} = \frac{2N}{2} = N, \tag{3}$$

so $\frac{N-1}{2}$ and $\frac{N+1}{2}$ are consecutive integers which sum to N. □

Example

21 can be written as

Example

21 can be written as 10+11.

Example

21 can be written as 10+11.

- This tells us that odd numbers can always be written as a sum of consecutive integers in *at least* one way.

Example

21 can be written as $10+11$.

- This tells us that odd numbers can always be written as a sum of consecutive integers in *at least* one way.

Fact

The average of a collection of numbers is given by the sum divided by the size of the collection. Therefore, the sum of a collection of numbers is equal to the average times the number of terms.

$$\text{sum} = \text{avg} \cdot (\text{number of terms})$$

Odd Factors

Proposition

If $N = td$ where d is odd and greater than 1, N can be written as a sum of consecutive integers.

Proof.

- center our sum around $\frac{d}{2}$, i.e. $\frac{d-1}{2}$ and $\frac{d+1}{2}$

Odd Factors

Proposition

If $N = td$ where d is odd and greater than 1, N can be written as a sum of consecutive integers.

Proof.

- center our sum around $\frac{d}{2}$, i.e. $\frac{d-1}{2}$ and $\frac{d+1}{2}$
- numbers will be chosen so that there are a total of $2t$ terms:

$$\underbrace{\cdots + \frac{d-1}{2}}_{\text{t terms}} + \underbrace{\frac{d+1}{2} + \cdots}_{\text{t terms}}.$$

Odd Factors

Proposition

If $N = td$ where d is odd and greater than 1, N can be written as a sum of consecutive integers.

Proof.

- center our sum around $\frac{d}{2}$, i.e. $\frac{d-1}{2}$ and $\frac{d+1}{2}$
- numbers will be chosen so that there are a total of $2t$ terms:

$$\underbrace{\cdots + \frac{d-1}{2}}_{\text{t terms}} + \underbrace{\frac{d+1}{2} + \cdots}_{\text{t terms}}.$$

- Check:

Odd Factors

Proposition

If $N = td$ where d is odd and greater than 1, N can be written as a sum of consecutive integers.

Proof.

- center our sum around $\frac{d}{2}$, i.e. $\frac{d-1}{2}$ and $\frac{d+1}{2}$
- numbers will be chosen so that there are a total of $2t$ terms:

$$\underbrace{\cdots + \frac{d-1}{2}}_{\text{t terms}} + \underbrace{\frac{d+1}{2} + \cdots}_{\text{t terms}}.$$

- Check: average of the terms above is equal to $\frac{d}{2}$

Odd Factors

Proposition

If $N = td$ where d is odd and greater than 1, N can be written as a sum of consecutive integers.

Proof.

- center our sum around $\frac{d}{2}$, i.e. $\frac{d-1}{2}$ and $\frac{d+1}{2}$
- numbers will be chosen so that there are a total of $2t$ terms:

$$\underbrace{\cdots + \frac{d-1}{2}}_{\text{t terms}} + \underbrace{\frac{d+1}{2} + \cdots}_{\text{t terms}}.$$

- Check: average of the terms above is equal to $\frac{d}{2}$
- The number of terms is $2t$

Odd Factors

Proposition

If $N = td$ where d is odd and greater than 1, N can be written as a sum of consecutive integers.

Proof.

- center our sum around $\frac{d}{2}$, i.e. $\frac{d-1}{2}$ and $\frac{d+1}{2}$
- numbers will be chosen so that there are a total of $2t$ terms:

$$\underbrace{\cdots + \frac{d-1}{2}}_{\text{t terms}} + \underbrace{\frac{d+1}{2} + \cdots}_{\text{t terms}}.$$

- Check: average of the terms above is equal to $\frac{d}{2}$
- The number of terms is $2t$
- The product of these is $\frac{d}{2}2t = td = N$, as expected.

Test

Example

$50 = 25$*2, so $d = 25$ and $t = 2$. The above algorithm then produces $50 = 11 + 12 + 13 + 14$.

Test

Example

$50 = 25*2$, so $d = 25$ and $t = 2$. The above algorithm then produces $50 = 11 + 12 + 13 + 14$.

Example

$50 = 5*10$, so $d = 5$ and $t = 10$. The above algorithm produces $50 = -7 + -6 + -5 + \cdots + 7 + 8 + 9 + 10 + 11 + 12$. If we wish to consider only positive numbers, then we can notice that $-7 + -6 + \cdots + 6 + 7 = 0$, so $50 = 8 + 9 + 10 + 11 + 12$.

Where are we?

- odd numbers can be written as sums of 2 consecutive integers
- if the number has an odd factor, it can be written as a sum of consecutive integers

Where are we?

- odd numbers can be written as sums of 2 consecutive integers
- if the number has an odd factor, it can be written as a sum of consecutive integers
- We are only left with numbers which do not have any odd factors - 2^n
- By a bit of experimentation, we hypothesize that these numbers cannot be written as a sum of consecutive integers

Proposition

If $N = 2^n$, then N cannot be written as a sum of consecutive integers.

Proof.

Suppose N can be written as a sum of consecutive integers, say $N = a + \cdots + b$. We will again use our fact.

- The average of a and b is $\frac{a+b}{2}$. The average of $a+1$ and $b-1$ is also $\frac{a+b}{2}$. The number of terms is $b-a+1$. We now know that

$$N = a + \cdots + b = \frac{a+b}{2}(b-a+1) = \frac{(a+b)(b-a+1)}{2}$$

Proposition

If $N = 2^n$, then N cannot be written as a sum of consecutive integers.

Proof.

Suppose N can be written as a sum of consecutive integers, say $N = a + \cdots + b$. We will again use our fact.

- The average of a and b is $\frac{a+b}{2}$. The average of $a+1$ and $b-1$ is also $\frac{a+b}{2}$. The number of terms is $b-a+1$. We now know that

$$N = a + \cdots + b = \frac{a+b}{2}(b-a+1) = \frac{(a+b)(b-a+1)}{2}$$

- If both a and b are even, $b-a+1$ must be odd
- If both a and b are odd, then $b-a+1$ is odd
- If a is even and b is odd, $a+b$ is odd.
- If a is odd and b is even, $a+b$ is odd.

Odd Factors

- No matter what the pairing, one of the factors of N is odd.

Odd Factors

- No matter what the pairing, one of the factors of N is odd.
- This means that *if* the number N can be written as a sum of consecutive integers, then it must have an odd factor. Since number of the form 2^n do not have any odd factors, they cannot be written as a sum of consecutive integers.

The new question

Given a number N, determine the number of ways that N can be written as a sum of consecutive integers.

We begin by determining which numbers can be written as a sum of 2 numbers, sum of 3 numbers, and so on. Consider the following table:

	2 numbers	3 numbers	4 numbers	5 numbers
1				
2				
3	1+2			
4				
5	2+3			
6		1+2+3		
7	3+4			
8				
9	4+5	2+3+4		
10			1+2+3+4	
⋮				
15	7+8	4+5+6		1+2+3+4+5

A new formula

- the smallest number which can be written as a sum of n number is $1 + 2 + \cdots + n$ (the triangular numbers)
- the formula for the n^{th} triangular number is $\frac{(n)(n+1)}{2}$

A new formula

- the smallest number which can be written as a sum of n number is $1 + 2 + \cdots + n$ (the triangular numbers)
- the formula for the n^{th} triangular number is $\frac{(n)(n+1)}{2}$
- for sum of 4 numbers: $10 + 4k$ where k is an integer. The 4 is because we are in the 4th column, and the 10 because it's the 4th triangular number.

A new formula

- the smallest number which can be written as a sum of n number is $1 + 2 + \cdots + n$ (the triangular numbers)
- the formula for the n^{th} triangular number is $\frac{(n)(n+1)}{2}$
- for sum of 4 numbers: $10 + 4k$ where k is an integer. The 4 is because we are in the 4th column, and the 10 because it's the 4th triangular number.
- The numbers which can be written as a sum of n numbers is then given by $\frac{n(n+1)}{2} + nk$ where k is an integer. We can easily write a computer program to check, then, whether numbers can be written as sums of consecutive integers.

Example

50 can be written as a sum of 5 numbers since

$$\begin{aligned} 50 &= \frac{5 * 6}{2} + 5k \\ &= 15 + 5k \\ \implies 35 &= 5k \\ \implies 7 &= k, \end{aligned}$$

which is an integer.

Example

50 can be written as a sum of 5 numbers since

$$\begin{aligned} 50 &= \frac{5*6}{2} + 5k \\ &= 15 + 5k \\ \implies 35 &= 5k \\ \implies 7 &= k, \end{aligned}$$

which is an integer. 50 cannot be written as a sum of 3 numbers since

$$\begin{aligned} 50 &= \frac{3*4}{2} + 3k \\ &= 6 + 3k \\ \implies 44 &= 3k \\ \implies k &= \frac{44}{3}, \end{aligned}$$

which is not an integer.

A new formula

We will temporarily forget the restriction that the numbers in the sums be positive. Remember - this will just double count everything.

A new formula

We will temporarily forget the restriction that the numbers in the sums be positive. Remember - this will just double count everything.

We will divide sums in to even and odd numbers of terms.

A new formula

We will temporarily forget the restriction that the numbers in the sums be positive. Remember - this will just double count everything.

We will divide sums in to even and odd numbers of terms.

If a number shows up as a sum of an even number of positive integers, what can we say about another sum if we allow negatives?

An even number of summands

Suppose N can be written as a sum of an even number of consecutive integers. Let

$$N = (m - n + 1) + \cdots + m + \cdots + (m + n) \tag{6}$$

What can we now say?

An even number of summands

Suppose N can be written as a sum of an even number of consecutive integers. Let

$$N = (m - n + 1) + \cdots + m + \cdots + (m + n) \tag{6}$$

What can we now say?

- The average is $\frac{(m+n)+(m-n+1)}{2} = \frac{2m+1}{2}$.
- The number of terms is $(m + n) - (m - n + 1) + 1 = 2n$.

An even number of summands

Suppose N can be written as a sum of an even number of consecutive integers. Let

$$N = (m - n + 1) + \cdots + m + \cdots + (m + n) \tag{6}$$

What can we now say?

- The average is $\frac{(m+n)+(m-n+1)}{2} = \frac{2m+1}{2}$.
- The number of terms is $(m + n) - (m - n + 1) + 1 = 2n$.
- The product, and thus N, is given by $(2m + 1)(n)$. Since $2m + 1$ is always odd, we can use this to help us.

An odd number of summands

Now suppose the number of terms in the sum is odd. Say,

$$N = (m - n) + \cdots + m + (m + 1) + \cdots + (m + n). \tag{7}$$

Again, we will use fact 2. The average is $\frac{(m+n)+(m-n)}{2} = m$. The number of terms is $(m + n) - (m - n) + 1 = 2n + 1$. N is thus equal to $(m)(2n + 1)$. Again, we have one term, $2n + 1$, which must be odd.

An example

Example

$N = 50 = 2 * 5^2$.

An example

Example

$N = 50 = 2 * 5^2$.

- If 50 is written as a sum of an even number of terms, $2m + 1$ could be 1, 5, or 25,

An example

Example

$N = 50 = 2 * 5^2$.

- If 50 is written as a sum of an even number of terms, $2m + 1$ could be 1, 5, or 25,
- which would make m 0, 2, or 12 respectively,

An example

Example

$N = 50 = 2 * 5^2$.

- If 50 is written as a sum of an even number of terms, $2m + 1$ could be 1, 5, or 25,
- which would make m 0, 2, or 12 respectively,
- and thus n is 50, 10, and 2 respectively.

An example

Example

$N = 50 = 2 * 5^2$.

- If 50 is written as a sum of an even number of terms, $2m + 1$ could be 1, 5, or 25,
- which would make m 0, 2, or 12 respectively,
- and thus n is 50, 10, and 2 respectively.
- These correspond to sums $-49 + -48 + \cdots + 0 + 1 + 2 + 49 + 50$, $-7 + -6 + \cdots + 2 + 3 + \cdots + 12$, and $11 + 12 + 13 + 14$.

An example

Example

$N = 50 = 2 * 5^2$.

- If 50 is written as a sum of an even number of terms, $2m + 1$ could be 1, 5, or 25,
- which would make m 0, 2, or 12 respectively,
- and thus n is 50, 10, and 2 respectively.
- These correspond to sums $-49 + -48 + \cdots + 0 + 1 + 2 + 49 + 50$, $-7 + -6 + \cdots + 2 + 3 + \cdots + 12$, and $11 + 12 + 13 + 14$.
- Note the first sum, when only considering positive terms, is just 50, so this is a trivial case. It is always true that $m = 0$ will correspond to this trivial case. The second sum is equivalent to $8 + 9 + 10 + 11 + 12$.

Example continued

Example

- If 50 is written as a sum of an odd number of terms, $2n + 1$ could be equal to 1, 5, or 25.

Example continued

Example

- If 50 is written as a sum of an odd number of terms, $2n+1$ could be equal to 1, 5, or 25.
- n is then 0, 2, or 12.

Example continued

Example

- If 50 is written as a sum of an odd number of terms, $2n + 1$ could be equal to 1, 5, or 25.
- n is then 0, 2, or 12.
- m is then 50, 10, or 2.

Example continued

Example

- If 50 is written as a sum of an odd number of terms, $2n + 1$ could be equal to 1, 5, or 25.
- n is then 0, 2, or 12.
- m is then 50, 10, or 2.
- These correspond to the sums 50, $8 + 9 + 10 + 11 + 12$, or $-10 + -9 + \cdots + 2 + \cdots 13 + 14$.

Example continued

Example

- If 50 is written as a sum of an odd number of terms, $2n + 1$ could be equal to 1, 5, or 25.
- n is then 0, 2, or 12.
- m is then 50, 10, or 2.
- These correspond to the sums 50, $8 + 9 + 10 + 11 + 12$, or $-10 + -9 + \cdots + 2 + \cdots 13 + 14$.
- This last sum is equivalent to $11 + 12 + 13 + 14$. This shows all of the ways that 50 can be written as a sum of consecutive integers.

Another Example

Example

$N = 45 = 3^2 * 5.$

Another Example

Example

$N = 45 = 3^2 * 5.$

All the odd factors excluding 1 are 3, 5, 9, 15, and 45, so there are 5 ways to write 45 as a sum of positive consecutive integers

Observations

- Note that since only odd factors affect the number of ways N can be written, factors of 2 never affect the number of ways N can be written.

Observations

- Note that since only odd factors affect the number of ways N can be written, factors of 2 never affect the number of ways N can be written.
- In particular, any number and double that number (or half of it, if it is even) will have the same number of decompositions.

Observations

- Note that since only odd factors affect the number of ways N can be written, factors of 2 never affect the number of ways N can be written.
- In particular, any number and double that number (or half of it, if it is even) will have the same number of decompositions.

Theorem

Suppose N is written in its prime decomposition as $N = 2^{n_0} p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$. The 2's, as mentioned, are irrelevant. There are $(n_1 + 1)(n_2 + 1) \cdots (n_k + 1)$ possible odd factors, which includes 1, so N can be written as a sum of consecutive positive integers in $[(n_1 + 1)(n_2 + 1) \cdots (n_k + 1)] - 1$ distinct ways.

The last example

Example

$21168 = 2^4 \cdot 3^3 \cdot 7^2$ can be written as a sum of positive consecutive integers in

The last example

Example

$21168 = 2^4 \cdot 3^3 \cdot 7^2$ can be written as a sum of positive consecutive integers in $(3+1)(2+1) - 1 = 4 \cdot 3 - 1 = 12 - 1 = 11$ distinct ways.