On Consecutive Sums

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One may consider sums of consecutive numbers, and which numbers can be written in this way. The question of which numbers can be addressed in this way and is discussed below. Following a discussion of which numbers can be written as consecutive sums, we determine an algorithm for determining how many distinct ways a given number can be written as a consecutive sum.

The issue of which numbers are "allowed" in our consecutive sums also needs to be addressed. In order for the question to be interesting, we shall restrict ourselves to integers. We can also see that allowing negative integers provides us with new information. This is discussed below, but an example will suffice to provide the idea. The number 9 can be written $2+3+4$, or as $-1+0+1+2+3+4$, but we see that these two are essentially the same sum.

1 Observations

Proposition 1. *Any odd number may be written as a sum of two consecutive integers.*

Proof. Consider an odd number N. Both $N+1$ and $N-1$ are even, so $\frac{N+1}{2}$ and $\frac{N-1}{2}$ are both integers. $\frac{N+1}{2}$ and $\frac{N-1}{2}$ are consecutive since

$$\frac{N-1}{2}+1=\frac{N-1}{2}+\frac{2}{2}=\frac{N+1}{2}. \tag{2}$$

Finally,

$$\frac{N-1}{2}+\frac{N+1}{2}=\frac{2N}{2}=N, \tag{3}$$

so $\frac{N-1}{2}$ and $\frac{N+1}{2}$ are consecutive integers which sum to N. □

Example 4. 21 can be written as 10+11.

This tells us that odd numbers can always be written as a sum of consecutive integers in *at least* one way.

Fact 5. The average of a collection of numbers is given by the sum divided by the size of the collection. Therefore, the sum of a collection of numbers is equal to the average times the number of terms.

Proposition 6. *If $N = td$ where d is odd and greater than 1, N can be written as a sum of consecutive integers.*

Proof. We center our sum around $\frac{d}{2}$, i.e. $\frac{d-1}{2}$ and $\frac{d+1}{2}$ will be the central numbers in our sum when ordered. The numbers will be chosen so that there are a total of $2t$ terms. We now have

$$\underbrace{\cdots + \frac{d-1}{2}}_{\text{t terms}} + \underbrace{\frac{d+1}{2} + \cdots}_{\text{t terms}}.$$

We will now use fact 5 to check that the sum given above is actually equal to $N = td$. The average of the terms above is equal to $\frac{d}{2}$. This can be seen by considering the average of the middle two terms, $\frac{d-1}{2}$ and $\frac{d+1}{2}$. The average of the terms immediately to the left and right of these two, $\frac{d+3}{2}$ and $\frac{d-3}{2}$, is also $\frac{d}{2}$. Since all of the numbers can be paired up in this way, the average of all the numbers is also $\frac{d}{2}$. The number of terms is $2t$. The product of these is $\frac{d}{2}2t = td = N$, as expected. □

Example 7. $50 = 25{*}2$, so $d = 25$ and $t = 2$. The above algorithm then produces $50 = 11 + 12 + 13 + 14$.

Example 8. $50 = 5{*}10$, so $d = 5$ and $t = 10$. The above algorithm produces $50 = -7 + -6 + -5 + \cdots + 7 + 8 + 9 + 10 + 11 + 12$. If we wish to consider only positive numbers, then we can notice that $-7 + -6 + \cdots + 6 + 7 = 0$, so $50 = 8 + 9 + 10 + 11 + 12$.

We now know that odd numbers can be written as sums of 2 consecutive integers. We also know that if the number has an odd factor (so can be written in the way above), it can be written as a sum of consecutive integers. We are only left with numbers which do not have any odd factors. By considering the prime factorization of such numbers, we can see that these numbers are of the form 2^n. By a bit of experimentation, we hypothesize that these numbers cannot be written as a sum of consecutive integers.

Proposition 9. *If $N = 2^n$, then N cannot be written as a sum of consecutive integers.*

Proof. This proof will be slightly more general than needed, but gives some hint of where we will go with the propositions later. Suppose N can be written as a sum of consecutive integers, say

$$N = a + \cdots + b.$$

We will again use fact 5. The average of a and b is $\frac{a+b}{2}$. The average of $a+1$ and $b-1$ is also $\frac{a+b}{2}$. If there are an even number of terms in this sum, then all numbers can be paired up this way, so the average of all the terms is $\frac{a+b}{2}$. If there are an odd number of terms, then all can be paired up except the center number, which is $\frac{a+b}{2}$, so the average of all the numbers is again $\frac{a+b}{2}$. The number of terms is $b - a + 1$. We now know that

$$N = a + \cdots + b = \frac{a+b}{2}(b - a + 1) = \frac{(a+b)(b-a+1)}{2} \tag{10}$$

Consider the possibilities for the parity (evenness or oddness) of a and b. If both a and b are even, $b-a+1$ must be odd. If both a and b are odd, then $b-a+1$ is odd. If a is even and b is odd, $a+b$ is odd. If a is odd and b is even, $a+b$ is odd. No matter what the pairing, one of the factors of N is odd. This means that *if* the number N can be written as a sum of consecutive integers, then it must have an odd factor. Since number of the form 2^n do not have any odd factors, they cannot be written as a sum of consecutive integers. □

We now have the beginnings of a way to analyze how numbers can be written as sums of consecutive integers. We now turn our attention to a more specific question. *Given a number N, determine the number of ways that N can be written as a sum of consecutive integers.*

2 Generalizations

We begin by determining which numbers can be written as a sum of 2 numbers, sum of 3 numbers, and so on. Consider the following table:

	2 numbers	3 numbers	4 numbers	5 numbers
1				
2				
3	1+2			
4				
5	2+3			
6		1+2+3		
7	3+4			
8				
9	4+5	2+3+4		
10			1+2+3+4	
⋮				
15	7+8	4+5+6		1+2+3+4+5

We can see that the smallest number which can be written as a sum of n number is $1+2+\cdots+n$. These numbers are called the triangular numbers, and the formula for the n^{th} triangular number is $\frac{(n)(n+1)}{2}$. The numbers in each column can be determined by considering a diagram. Consider the numbers which can be written as a sum of four numbers. We begin with 1+2+3+4:

.
. .
. . .
. . . .

The next sum of four number is 2+3+4+5:

. .

. . .
. . . .
. . . ..
or $1+2+3+4+4$. Next will be $1+2+3+4+4+4$. In general, this will be $10+4k$ where k is an integer. The four is because we are in the 4th column, and the 4 because it's the 4th triangular number. The numbers which can be written as a sum of n numbers is then given by $\frac{n(n+1)}{2}+nk$ where k is an integer. We can easily write a computer program to check, then, whether numbers can be written as sums of consecutive integers.

Example 11. 50 can be written as a sum of 5 numbers since

$$\begin{aligned} 50 &= \frac{5*6}{2}+5k \\ &= 15+5k \\ \implies 35 &= 5k \\ \implies 7 &= k, \end{aligned}$$

which is an integer. 50 cannot be written as a sum of 3 numbers since

$$\begin{aligned} 50 &= \frac{3*4}{2}+3k \\ &= 6+3k \\ \implies 44 &= 3k \\ \implies k &= \frac{44}{3}, \end{aligned}$$

which is not an integer.

3 Further Generality

We will now consider the problem in even more generality. We will temporarily forget the restriction that the numbers in the sums be positive.

Suppose N can be written as a sum of an even number of consecutive integers. Let

$$N = (m-n+1)+\cdots+m+\cdots+(m+n) \tag{12}$$

What can we now say? Again using fact 5, the sum is equal to the average times the number of terms. The average is $\frac{(m+n)+(m-n+1)}{2} = \frac{2m+1}{2}$. The number of terms is $(m+n)-(m-n+1)+1=2n$. The product, and thus N, is given by $(2m+1)(n)$. Since $2m+1$ is always odd, we can use this to help us.

Now suppose the number of terms in the sum is odd. Say,

$$N = (m-n)+\cdots+m+(m+1)+\cdots+(m+n). \tag{13}$$

Again, we will use fact 5. The average is $\frac{(m+n)+(m-n)}{2} = m$. The number of terms is $(m+n)-(m-n)+1=2n+1$. N is thus equal to $(m)(2n+1)$. Again, we have one term, $2n+1$, which must be odd.

Example 14. Consider $N = 50 = 2 * 5^2$. If 50 is written as a sum of an even number of terms, $2m+1$ could be 1, 5, or 25, which would make m 0, 2, or 12 respectively, and thus n is 50, 10, and 2 respectively. These correspond to sums $-49 + -48 + \cdots + 0 + 1 + 2 + 49 + 50$, $-7 + -6 + \cdots + 2 + 3 + \cdots + 12$, and $11 + 12 + 13 + 14$. Note the first sum, when only considering positive terms, is just 50, so this is a trivial case. It is always true that $m = 0$ will correspond to this trivial case. The second sum is equivalent to $8 + 9 + 10 + 11 + 12$.

If 50 is written as a sum of an odd number of terms, $2n + 1$ could be equal to 1, 5, or 25. n is then 0, 2, or 12. m is then 50, 10, or 2. These correspond to the sums 50, $8 + 9 + 10 + 11 + 12$, or $-10 + -9 + \cdots + 2 + \cdots 13 + 14$. This last sum is equivalent to $11 + 12 + 13 + 14$. This shows all of the ways that 50 can be written as a sum of consecutive integers.

Notice that every sum can be written in two ways when we allow negative numbers. If the sum begins with 1, we could also include 0. If the sum begins with k, we could begin with $-(k-1)$ to $k-1$ as well. Each option adds an odd number of terms, so if one version of the sum had an odd number of terms, it could have an even number with the negatives tacked on an vice versa. This means that if we restrict ourselves to only positive numbers, we have double counted. Since the $2m+1 = 1$ case for even numbers of terms and $2n+1 = 1$ case for odd number of terms are always trivial, the odd factors of N (excluding 1) will count the number of ways N can be written as a sum of consecutive positive integers. The odd factors would be counted in each category of odd and even numbers of terms, but this must be divided by two to correct for the double counting.

Example 15. $N = 45 = 3^2 * 5$. All the odd factors excluding 1 are 3, 5, 9, 15, and 45, so there are 5 ways to write 45 as a sum of positive consecutive integers.

Note that since only odd factors affect the number of ways N can be written, factors of 2 never affect the number of ways N can be written. In particular, any number and double that number (or half of it, if it is even) will have the same number of decompositions.

Finally, the last generalization can be made. Suppose N is written in its prime decomposition as $N = 2^{n_0} p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$. The 2's, as mentioned, are irrelevant. There are $(n_1 + 1)(n_2 + 1) \cdots (n_k + 1)$ possible odd factors, which includes 1, so N can be written as a sum of consecutive positive integers in $[(n_1 + 1)(n_2 + 1) \cdots (n_k + 1)] - 1$ distinct ways.

Example 16. $21168 = 2^4 \cdot 3^3 \cdot 7^2$ can be written as a sum of positive consecutive integers in $(3 + 1)(2 + 1) - 1 = 4 \cdot 3 - 1 = 12 - 1 = 11$ distinct ways.