Approximation of the Dispersionless Toda Lattice by Toeplitz Operators

by

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Chapter 1
Introduction

The Toda lattice is a model for a system of particles with a nearest-neighbor interaction. One can model such a system with a potential function of the form $V(r) = \frac{r^2}{2}$. In 1955, Enrico Fermi, John Pasta, and Stanislaw Ulam conducted experiments investigating this type of model to understand a 1-dimensional crystal. Morikazu Toda investigated a variation of this interaction law using $V(r) = e^{-r} + r - 1$ as a potential function, which now bears his name. One especially appealing aspect to Toda's system is the existence of solutions which do not change shape or size over time. These waves, called solitons, have become a significant subject of research. Another related equation, the Korteweg de Vries equation, exhibits soliton solutions as well, and is closely related to the Toda lattice.

One can investigate a particular limiting behavior of the Toda lattice, called the dispersionless Toda lattice. This analysis produces a system of partial differential equations, which have been studied. In particular, a recent result has been found by Bloch, Golse, Paul, and Uribe ([BGPU03]). Their result is the motivation for this paper. Here, we outline the necessary background to understand this result and provide the ideas of the proofs. Since the actual proofs are somewhat technical, sketches will be provided in the last sections. The reader is directed to [BGPU03] for the details of the proofs.

Bloch, Golse, Paul, and Uribe's investigations heavily use a specific type of operator called the Toeplitz operator. These operators will be used to establish bounds on solutions of the Toda lattice equations.

The background material begins with an overview of Hamiltonian mechanics and the symplectic geometry required to do Hamiltonian mechanics. At this stage we provide the formal definition of the Toda lattice. We will also describe a process called geometric quantization which provides much of the machinery which we will need. Specific boundary conditions on the Toda lattice will force us to consider the quantization of two particular manifolds - the sphere and the torus. To do this, we must build up the machinery of connections, curvature, line bundles, and cohomology. Once this is done, the quantizations can be carried out and the Toeplitz operators constructed. As an aside, we also discuss the concept of dispersion, particularly in the case of the KdV equation. The paper concludes with an overview of the ideas and sketches of the proofs in the paper by Bloch, Golse, Paul, and Uribe.

Chapter 2
Hamiltonian Mechanics and Symplectic Geometry

2.1 Hamiltonian Mechanics

Motivation 2.1.1. The Toda lattice is a model for a collection of identical masses connected linearly by springs. The formal way to describe this system is by way of Hamiltonian mechanics.

Definition 2.1.2. A *Hamiltonian function* is a smooth function

$$\mathcal{H} : (q^1, q^2, \dots, q^n, p_1, p_2, \dots, p_n) \to \mathbb{R}, \tag{2.1.3}$$

The q^i coordinate represents the position of the i^{th} mass, and the p_i coordinate represents the momentum of the i^{th} mass.

Remark 2.1.4. The Hamiltonian function often represents the energy of a system.

Remark 2.1.5. We will begin using *Einstein summation notation* now. This means that any time an index appears as a subscript and superscript, there is an implied summation over that index.

Definition 2.1.6 (Hamiltonian vector field)**.** For a given Hamiltonian function $\mathcal{H}$, the *Hamiltonian vector field* is defined by

$$\mathcal{X}_{\mathcal{H}} = \frac{\partial \mathcal{H}}{\partial p_i} dq^i - \frac{\partial \mathcal{H}}{\partial q^i} dp_i = \left(\frac{\partial \mathcal{H}}{\partial p_1}, \dots \frac{\partial \mathcal{H}}{\partial p_n}, -\frac{\partial \mathcal{H}}{\partial q^1}, \dots, -\frac{\partial \mathcal{H}}{\partial q^n} \right) \tag{2.1.7}$$

Remark 2.1.8. The Hamiltonian vector field has a very natural interpretation. The set of points $(q^1, q^2, \dots, q^N, p_1, p_2, \dots, p_N)$ is called the *phase space* for a classical mechanical system. This space represents all possible combinations of position and momentum for the masses in the given system. The *configuration space* is the set of position coordinates, $(q^1, q^2, \ldots, q^n)$. An integral path for the Hamiltonian vector field determines the path followed in time by the system. This vector field allows the time derivatives of position and momentum to be written in terms of only the Hamiltonian.

Definition 2.1.9 (Hamiltonian equations)**.** Given a Hamiltonian function, the time derivatives of position and momentum (denoted with a dot) can be computed using the *Hamiltonian equations*

$$\begin{cases} \dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i}. \end{cases} \tag{2.1.10}$$

Example 2.1.11 (Particle in a Potential field)**.** For a single particle with potential function $V(q)$, the Hamiltonian function is of the form

$$H(q,p) = \frac{p^2}{2m} + V(q). \tag{2.1.12}$$

The Hamiltonian equations are

$$\begin{cases} \dot{q} &= \frac{p}{m} \\ \dot{p} &= -\nabla V. \end{cases} \tag{2.1.13}$$

These two equations are the form Newton's laws take in Hamiltonian mechanics.

Example 2.1.14 (Simple Harmonic Oscillator)**.** The simple harmonic oscillator is a special case of (2.1.11) in which $V(q) = \frac{kq^2}{2}$. The computation of the Hamiltonian equations leads to the usual definition of momentum,

$$p = m\dot{q} \tag{2.1.15}$$

and

$$\dot{p} = -kq. \tag{2.1.16}$$

The second equation is the standard equation of motion for a simple harmonic oscillator, since $\dot{p} = ma$. Setting k and m to be 1, which can be done by the appropriate choice of units, the hamiltonian function is

$$H = \frac{1}{2}q^2 + \frac{1}{2}p^2. \tag{2.1.17}$$

The hamiltonian vector field is easily computed since

$$\mathcal{X}_H = (\dot{q}, \dot{p}) = (p, -q). \tag{2.1.18}$$

This flow is just clockwise rotation in the (q, p) plane.

Definition 2.1.19 (Poisson bracket)**.** Given two functions f and g, each of q^i and p_i, the *Poisson bracket* of f and g is given by

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}. \tag{2.1.20}$$

Remark 2.1.21. The Poisson bracket allows the time derivative of an arbitrary function F to be computed along solution curves of (2.1.7) by

$$\dot{F} = \{F, \mathcal{H}\}, \tag{2.1.22}$$

since

$$\frac{dF}{dt} = \sum_i \left(\frac{\partial F}{\partial q^i}\,\dot{q}^i + \frac{\partial F}{\partial p_i}\,\dot{p}_i\right) = \sum_i \left(\frac{\partial F}{\partial q^i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q^i}\right) = \{F, \mathcal{H}\}. \tag{2.1.23}$$

The Poisson bracket is antisymmetric, so in particular,

$$\dot{\mathcal{H}} = \{\mathcal{H}, \mathcal{H}\} = 0. \tag{2.1.24}$$

Therefore, the energy of the system is conserved.

2.2 Definition of The Toda Lattice

Definition 2.2.1 (The Toda Lattice)**.** Consider N identical masses connected by identical springs in a 1-dimensional chain. Units may be chosen to normalize the mass $m = 1$ and the resistance of the system $k = 1$. A Hamiltonian modelling the nonlinear nearest-neighbor interactions is defined by

$$\mathcal{H}(q^1, q^2, \dots, q^N, p_1, p_2, \dots, p_N) = \frac{1}{2}\sum_{j=1}^{N} p_i^2 + \sum_{j=1}^{N-1} e^{q^j - q^{j+1}}. \tag{2.2.2}$$

This is called the *Toda Lattice*. The kinetic energy is the standard $\frac{1}{2}\sum_{j=1}^{N} p_i^2$, and the potential energy is $\sum_{j=1}^{N-1} e^{q^j - q^{j+1}}$. The associated Hamiltonian equations are

$$\begin{cases} \dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i} = p_i, \ i = 1, \dots, N \\ \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i} = e^{q^{i-1} - q^i} - e^{q^i - q^{i+1}}, \ i = 2, \dots, N-1 \\ \dot{p}_1 = -\frac{\partial \mathcal{H}}{\partial q^1} = -e^{q^1 - q^2} \\ \dot{p}_N = -\frac{\partial \mathcal{H}}{\partial q^N} = e^{q^{N-1} - q^N} \end{cases} \tag{2.2.3}$$

Remark 2.2.4. The construction of the Hamiltonian for the Toda lattice could be approached in a slightly different way. If $V(r) = e^{-r} + r - 1$ is the potential function, the Hamiltonian function is defined

$$\begin{aligned} \mathcal{H}(q^1, q^2, \dots, q^N, p_1, p_2, \dots, p_N) &= \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i=1}^{N-1} V(q^{i+1} - q^i) \qquad (2.2.5) \\ &= \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i=1}^{N-1} \left(e^{q^i - q^{i+1}} + q^{i+1} - q^i - 1 \right) \\ &= q^N - q^1 + \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i=1}^{N-1} e^{q^i - q^{i+1}} - 1. \end{aligned}$$

The potential $V(r) = e^{-r} + r - 1$ would be appropriate because the Taylor series approximation is $V(r) \approx \frac{r^2}{2}$, which is the harmonic potential. The equations of motion for this Hamiltonian are

$$\begin{cases} \dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i} = p_i \text{ for } i = 1, \dots, N \\ \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i} = e^{q^{i-1} - q^i} - e^{q^i - q^{i+1}} \text{ for } i = 2, \dots, N-1 \\ \dot{p}_1 = -\frac{\partial \mathcal{H}}{\partial q^1} = 1 - e^{q^1 - q^2} \\ \dot{p}_N = -\frac{\partial \mathcal{H}}{\partial q^N} = -1 + e^{q^{N-1} - q^N} \end{cases} \tag{2.2.6}$$

These equations are nearly the same as those of the original definition of the Hamiltonian, (2.2.3). We can then consider (2.2.2) to be the Hamiltonian equation for the Toda lattice.

Proposition 2.2.7. *By applying the change of coordinates*

$$\begin{cases} a_j = \frac{1}{2} e^{\frac{q^j - q^{j+1}}{2}} \\ b_j = -\frac{p_j}{2}, \end{cases} \tag{2.2.8}$$

the following relations hold:

$$\{a_n, b_n\} = -\frac{1}{4} a_n \tag{2.2.9}$$

$$\{a_n, b_{n+1}\} = \frac{1}{4} a_n \tag{2.2.10}$$

$$\dot{a}_n = a_n(b_{n+1} - b_n) \tag{2.2.11}$$

$$\dot{b}_n = 2(a_n^2 - a_{n-1}^2). \tag{2.2.12}$$

Proof. Begin by noticing

$$\begin{cases} \{p_i, q^i\} = -1 \\ \{q^i, p_i\} = 1 \end{cases} \tag{2.2.13}$$

immediately by the definition of the Poisson bracket. The only non-zero brackets in the new coordinates can be computed:

$$\{a_n, b_n\} = \{\frac{1}{2} e^{\frac{q^n - q^{n+1}}{2}}, -\frac{p_n}{2}\} \tag{2.2.14}$$

$$= \sum_i \left[\frac{\partial}{\partial q^i} \left(\frac{1}{2} e^{\frac{q^n - q^{n+1}}{2}} \right) \cdot \frac{\partial}{\partial p_i} \left(-\frac{p_n}{2} \right) - \frac{\partial}{\partial p_i} \left(\frac{1}{2} e^{\frac{q^n - q^{n+1}}{2}} \right) \cdot \frac{\partial}{\partial q^i} \left(-\frac{p_j}{2} \right) \right] \tag{2.2.15}$$

$$= \frac{1}{4} e^{\frac{q^n - q^{n+1}}{2}} \cdot -\frac{1}{2} = -\frac{1}{4} a_n \tag{2.2.16}$$

and

$$\{a_n, b_{n+1}\} = \{\frac{1}{2}e^{\frac{q^n - q^{n+1}}{2}}, -\frac{p_{n+1}}{2}\} \tag{2.2.17}$$

$$= \sum_i \frac{\partial}{\partial q^i}\left(\frac{1}{2}e^{\frac{q^n - q^{n+1}}{2}}\right) \cdot \frac{\partial}{\partial p_i}\left(-\frac{p_{n+1}}{2}\right) \tag{2.2.18}$$

$$- \frac{\partial}{\partial p_i}\left(\frac{1}{2}e^{\frac{q^n - q^{n+1}}{2}}\right) \cdot \frac{\partial}{\partial q^i}\left(-\frac{p_{n+1}}{2}\right) \tag{2.2.19}$$

$$= -\frac{1}{4}e^{\frac{q^n - q^{n+1}}{2}} \cdot -\frac{1}{2} = \frac{1}{4}a_n. \tag{2.2.20}$$

The time derivatives can also be computed directly. We can only consider the terms which are well-defined without boundary conditions. For $2 \leq j \leq N-1$,

$$\begin{aligned} \dot{a}_n &= \frac{d}{dt}\frac{1}{2}e^{\frac{q^n - q^{n+1}}{2}} \\ &= \frac{1}{4}e^{\frac{q^n - q^{n+1}}{2}} \cdot \dot{q}^n - \dot{q}^{n+1} \\ &= \frac{1}{4}e^{\frac{q^n - q^{n+1}}{2}} \cdot p_n - p_{n+1} \\ &= a_n(b_{n+1} - b_n) \end{aligned} \tag{2.2.21}$$

and

$$\begin{aligned} \dot{b}_n &= \frac{d}{dt} - \frac{p_n}{2} \\ &= -\frac{1}{2}e^{q^{n-1} - q^i} - e^{q^n - q^{n+1}} \\ &= -2(a_{n-1}^2 - a_n^2) \\ &= 2(a_n^2 - a_{n-1}^2). \end{aligned} \tag{2.2.22}$$

□

Motivation 2.2.23. There is an indeterminacy in the change of coordinates we have chosen. To determine solutions to the differential equations, we need to know the values of a_0 and a_N. In the q and p coordinates, choosing q^0 and q^{N+1} is equivalent to choosing a_0 and a_N. Two particular choices for these values will be considered.

Definition 2.2.24. If a_0 and a_N are both chosen to be 0, the lattice is called the *non-periodic Toda lattice*. This condition corresponds to formally setting $q_0 = -\infty$ and $q_{N+1} = +\infty$. If the boundary conditions are fixed by choosing $a_{j+N} = a_j$ and $b_{j+N} = b_j$, the lattice is called the *periodic Toda lattice*.

Remark 2.2.25. One can define a quasi-periodic boundary condition for the Toda lattice, given by $a_N = e^{-2\pi\nu}a_0$, though this will not be considered.

2.3 Matrix Description of the Toda Lattice

Definition 2.3.1. A *Lax pair* is a pair of matrices $L(t)$ and $B(L(t))$ such that

$$\frac{dL}{dt} = [B(L(t)), L(t)]. \tag{2.3.2}$$

Motivation 2.3.3. The change of coordinates into the a and b coordinates above allow (2.2.21) and (2.2.22) to be written as a Lax pair with the following matrices.

In the periodic case,

$$L(t) = \begin{pmatrix} b_1 & a_1 & 0 & \dots & a_N \\ a_1 & b_2 & a_2 & \dots & 0 \\ 0 & a_2 & b_3 & a_3 \dots & 0 \\ \dots & \dots & & \dots & \vdots \\ a_N & \dots & & a_{N-1} & b_N \end{pmatrix} \tag{2.3.4}$$

and

$$B(L(t)) = \begin{pmatrix} 0 & a_1 & 0 & \dots & -a_N \\ -a_1 & 0 & a_2 & \dots & 0 \\ 0 & -a_2 & 0 & a_3 \dots & 0 \\ \dots & \dots & & \dots & \vdots \\ a_N & \dots & & -a_{N-1} & 0 \end{pmatrix}. \tag{2.3.5}$$

In the non-periodic case,

$$L(t) = \begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 \\ a_1 & b_2 & a_2 & \dots & 0 \\ 0 & a_2 & b_3 & a_3 \dots & 0 \\ \dots & \dots & & \dots & \vdots \\ 0 & \dots & & a_{N-1} & b_N \end{pmatrix} \tag{2.3.6}$$

and

$$B(L(t)) = \begin{pmatrix} 0 & a_1 & 0 & \dots & 0 \\ -a_1 & 0 & a_2 & \dots & 0 \\ 0 & -a_2 & 0 & a_3 \dots & 0 \\ \dots & \dots & & \dots & \vdots \\ 0 & \dots & & -a_{N-1} & 0 \end{pmatrix}. \tag{2.3.7}$$

For the non-periodic lattice, $B(L)$ can be written as $B(L) = [L, \mathcal{N}]$, where

$$\mathcal{N} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 3 & \dots & 0 \\ \dots & \dots & & \dots & \vdots \\ 0 & \dots & & 0 & N \end{pmatrix}. \tag{2.3.8}$$

Definition 2.3.9. An *integral of motion* is a function f such that $\{f, H\} = 0$.

Definition 2.3.10. If $\mathcal{H}(q^1, q^2, \dots, q^n, p_1, p_2, \dots, p_n)$ is a Hamiltonian function for a system, then the Hamiltonian system is called *integrable* or *completely integrable* if there exist n integrals of motion I_j such that $\{I_j, I_k\} = 0 \; \forall j \neq k$ and $dI_1 \wedge dI_2 \wedge \dots \wedge dI_n \neq 0$.

Remark 2.3.11. The Hamiltonian itself will be an integral of motion (2.3.9). For the Toda lattice, the eigenvalues of the matrix L (assumed distinct) are integrals of motion, since they are unchanged over time, as we now show.

Proposition 2.3.12. *The eigenvalues of a matrix satisfying* (2.3.2) *are constant with respect to time.*

Proof. Consider an eigenvalue λ with unit eigenvector v.

$$
\begin{aligned}
Lv &= \lambda v \\
\Rightarrow L_t v + L v_t &= \lambda_t v + \lambda v_t \\
\Rightarrow \lambda_t v &= L_t v + L v_t - \lambda v_t \\
&= BLv - LBv + (L - \lambda) v_t \\
&= B\lambda v - LBv + (L - \lambda) v_t \\
&= (\lambda - L) Bv + (L - \lambda) v_t \\
&= (L - \lambda)(v_t - Bv) \\
\Rightarrow \lambda_t (v, v) &= ((L - \lambda)(v_t - Bv), v) \\
&= ((v_t - Bv), (L - \lambda) v) \\
&= ((v_t - Bv), 0) = 0 \\
\Rightarrow \lambda_t &= 0.
\end{aligned}
$$

□

Corollary 2.3.13. *The periodic and non-periodic Toda lattices are both integrable systems.*

Proof. While the complete proof will not be given, we will give some information that is relevant and interesting to this system's integrability. For a complete proof, the reader is directed to [Fla74].

We will consider the non-periodic lattice first. The eigenvalues of the matrix L provide N values which are potential integrals of motion. This can be seen because, for an eigenvalue λ, $\{\lambda, H\} = \frac{d\lambda}{dt} = 0$. This means that the eigenvalues are at least first integrals, meaning they commute with the Hamiltonian. For integrability, we would need the eigenvalues to commute with each other. The non-periodic Toda lattice has conditions $a_0 = a_N = 0$, so there are $N - 1$ distinct values for $\{a_j\}$. Also, $\sum_i b_i$ is constant (momentum is conserved), so there exist $N - 1$ distinct values for

$\{b_j\}$. For the system to be integrable, we require $N-1$ integrals of motion. The eigenvalues of the matrix suffice to provide these values because the N eigenvalues have the constraint that their sum is $\sum_i b_i$, so there are $N-1$ linearly independent eigenvalues.

For the periodic Toda lattice, we must consider the N distinct values of a_j and the N distinct values of b_j. Recall the condition for the periodic Toda lattice is $a_j = a_{N+j}$ and $b_j = b_{N+j}$. We again have $\sum_i b_i$. We can also see that $\Pi_i a_i$ is constant since

$$\frac{d}{dt}\Pi_i a_i = \left(\prod_i a_i\right)\left(\sum_i \frac{\dot{a}_i}{a_i}\right) = 0. \tag{2.3.14}$$

□

2.4 First Discussion of The Result of Bloch, Golse, Paul, and Uribe

Motivation 2.4.1. The main goal of this paper is to understand the relationship between the system of $2N$ ordinary differential equations which define the Toda lattice and a particular pair of partial differential equations. We begin by considering two continuous functions on the unit interval, a and b. We will fixed particular values of these functions by demanding that they satisfy conditions related to the Toda lattice. Let

$$\begin{cases} a(\frac{j}{N}) = a_j \\ b(\frac{j}{N}) = b_j. \end{cases} \tag{2.4.2}$$

The spacing between the fixed values is $\frac{1}{N}$. If we then allow the functions a and b to vary over time, t, we can then require that a_j^t and b_j^t satisfy (2.2.11) and (2.2.12).

Definition 2.4.3. Define the variables

$$\begin{cases} x := \frac{j}{N} \\ s := \frac{t}{N}. \end{cases} \tag{2.4.4}$$

Proposition 2.4.5. *Let a_j^t and ${}_j^t$ be defined as above, satisfying the relations* (2.2.11) *and* (2.2.12)*. These relations, under the change of variables above and in the limit as $N \to \infty$ produce the partial differential equations*

$$\begin{cases} \partial_s a^s(x) = a^s(x)\partial_x b^s(x) \\ \partial_s b^s(x) = 2\partial_x (a^s(x))^2. \end{cases} \tag{2.4.6}$$

This is called the dispersionless limit of the Toda lattice*.*

Proof. We re-investigate the Hamiltonian equations in a and b, (2.2.21) and (2.2.22). Applying only the first change of variables, we see

$$\dot{a}_j^t = \dot{a}^t(\frac{j}{N}) = \dot{a}^t(x). \tag{2.4.7}$$

Likewise, $a_j^t = a^t(x)$. Furthermore,

$$b_{j+1}^t - b_j^t = b^t(\frac{j+1}{N}) - b^t(\frac{j}{N}) = b^t(x + \frac{1}{N}) - b^t(x). \tag{2.4.8}$$

We also have

$$\dot{b}_j^t = \frac{\partial}{\partial t} b^t(\frac{j}{N}) = \dot{b}^t(x) \tag{2.4.9}$$

and

$$2(\frac{\partial}{\partial t}(a_j^t)^2 - \frac{\partial}{\partial t}(a_{j-1}^t)^2) = 2((a^t(x))^2 - a^t(x - \frac{1}{N}))^2. \tag{2.4.10}$$

Putting these pieces together, we can see that in the limit as $N \to \infty$, the right hand sides of the equations

$$\begin{cases} \dot{a}_j^t = a_n^t(b_{n+1}^t - b_n^t) \\ \dot{b}_j^t = 2((a_j^t)^2 - (a_{j-1}^t)^2 \end{cases} \tag{2.4.11}$$

produce

$$N\partial_t a^t(x) = a^t \partial_x b^t(x) \tag{2.4.12}$$

and

$$N\partial_t b^t(x) = 2\partial_x (a^t(x)^2) \tag{2.4.13}$$

We then make the second change of variables, $s := \frac{t}{N}$. This allows us to deal with the coefficient of N in the left hand sides of the equations, since

$$\frac{\partial}{\partial t} = \frac{ds}{dt}\frac{\partial}{\partial s} = \frac{1}{N}\frac{\partial}{\partial s} \tag{2.4.14}$$

Terms of order $\frac{1}{N}$ vanish when in the limit $N \to \infty$, so (2.2.21) and (2.2.22) can be written

$$\begin{cases} \partial_s a^s(x) = a^s(x)\partial_x b^s(x) \\ \partial_s b^s(x) = 2\partial_x (a^s(x))^2. \end{cases} \tag{2.4.15}$$

□

Proposition 2.4.16. *This system is hyperbolic.*

Proof. Dropping superscripts and subscripts, rewrite the system (2.4.15) as

$$\begin{bmatrix} a \\ b \end{bmatrix}_s + \begin{bmatrix} 0 & -4b \\ -b & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}_x = 0. \tag{2.4.17}$$

The matrix in the above equation is diagonalizable as

$$\begin{bmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & -4b \\ -b & 0 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2b & 0 \\ 0 & -2b, \end{bmatrix} \tag{2.4.18}$$

with real eigenvalues. □

Remark 2.4.19. Hyperbolicity implies the system may develop shocks, depending on the initial conditions. This can be seen by imposing the initial conditions $b^{t=0} = 2a^{t=0}$, which reduces to Burger's equation:

$$\partial_s a = \partial_x a^2, \tag{2.4.20}$$

which is known to develop shocks.

Remark 2.4.21. More of the ideas of the Toda lattice can be found in [Blo03] and [DM98].

2.5 Symplectic Geometry

Motivation 2.5.1. Our goal is the quantization of symplectic manifolds, so we will review some ideas from symplectic geometry. The symplectic structure of a manifold allows us to formulate the Hamiltonian picture of classical mechanics.

Definition 2.5.2. A *symplectic manifold*, (M, ω) is a smooth manifold, M, equipped with a nondegenerate skew-symmetric, closed 2-form, ω, called the *symplectic form*.

Remark 2.5.3. We will prefer coordinate-independent formulations of statements, but the form could be written in local coordinates. In this case, ω will be written as a matrix at each point in the manifold, ω^{ij}. Nondegeneracy means that ω^{ij} has nonzero determinant. Skew symmetry means that $\omega^{ij} = -\omega^{ji}$, and the condition that ω is closed means $\frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{jk}}{\partial x^i} + \frac{\partial \omega_{ki}}{\partial x^j} = 0$.

Motivation 2.5.4. Suppose $\mathcal{H}$ is a hamiltonian function defined on the symplectic manifold (M, ω), so that $d\mathcal{H}$ is a 1-form on the manifold. The relation between $d\mathcal{H}$, ω, and the hamiltonian vector field can be constructed in a coordinate independent way. The definition presented here will be related back to the definition presented in (2.1.7) shortly.

Definition 2.5.5. The *Hamiltonian vector field*, denoted $X_{\mathcal{H}}$, is defined by the condition $\iota_{X_{\mathcal{H}}}\omega = d\mathcal{H}$.

Remark 2.5.6. The existence of the Hamiltonian vector field can be seen by considering the mapping of vector fields to one-forms

$$X \to \iota_X \omega = \omega(X, \cdot) \tag{2.5.7}$$

where ι represents the contraction operation. This map also gives an isomorphism between the tangent bundle and cotangent bundle of a symplectic manifold. Since ω is nondegenerate, $\omega(X, \cdot) = 0$, forces $X = 0$. This guarantees that the map is injective and surjective.

Motivation 2.5.8. Assuming that ω is only a 2-form on the manifold M satisfying the condition of (2.5.5), the properties of a symplectic form can be motivated by physical considerations to see why symplectic geometry is a natural setting for Hamiltonian mechanics.

1. Given a Hamiltonian function $\mathcal{H}$ and 2-form ω, $\iota(X_{\mathcal{H}})\omega = d\mathcal{H}$ should be solvable, so ω should be nondegenerate.

2. The flow on M generated by the Hamiltonian vector field should leave ω invariant. We would then expect the Lie derivative $\mathcal{L}_{X_{\mathcal{H}}}\omega$ to be zero. Evaluating using Cartan's formula,

$$\begin{aligned}\mathcal{L}_{X_{\mathcal{H}}}\omega &= d\iota_{X_{\mathcal{H}}}\omega + \iota_{X_{\mathcal{H}}}d\omega \\ &= dd\mathcal{H} + \iota_{X_{\mathcal{H}}}d\omega = \iota_{X_{\mathcal{H}}}d\omega.\end{aligned} \tag{2.5.9}$$

   Since ω is nondegenerate by the previous item, ω will be preserved if and only if ω is closed.

3. H should be invariant under the flow of $\mathcal{X}_H$. This can be checked

$$\mathcal{L}_{X_{\mathcal{H}}}(\mathcal{H}) = \mathcal{X}_{\mathcal{H}}(\mathcal{H}) = d\mathcal{H}(\mathcal{X}_{\mathcal{H}}) = \omega(\mathcal{X}_{\mathcal{H}}, \mathcal{X}_{\mathcal{H}}), \tag{2.5.10}$$

   which will be zero since ω is a 2-form, so skew-symmetric. The skew symmetry of ω then implies conservation of energy.

Example 2.5.11 (Height function on $\mathbb{S}^2$)**.** We can also consider $\mathbb{S}^2$ to be a phase space since it is a symplectic manifold. In a coordinate patch, $\mathbb{S}^2$ has a natural symplectic form $\omega = \sin(\theta)d\theta \wedge d\phi$, inherited by its embedding in $\mathbb{R}^3$. For our Hamiltonian, we will choose the height function

$$H(\theta, \phi) = \cos(\theta). \tag{2.5.12}$$

We will denote the Hamiltonian flow by

$$\mathcal{X}_H = \alpha\frac{\partial}{\partial\theta} + \beta\frac{\partial}{\partial\phi} \tag{2.5.13}$$

and a general vector by

$$v = a\frac{\partial}{\partial\theta} + b\frac{\partial}{\partial\phi}. \tag{2.5.14}$$

It follows that

$$dH = -\sin(\theta)d\theta \text{ so} \tag{2.5.15}$$

$$\omega(\mathcal{X}_H, v) = dH(v) \tag{2.5.16}$$

$$(\sin(\theta)d\theta \wedge d\phi)\left(\alpha\frac{\partial}{\partial\theta} + \beta\frac{\partial}{\partial\phi}, a\frac{\partial}{\partial\theta} + b\frac{\partial}{\partial\phi}\right) = -\sin(\theta)d\theta\left(a\frac{\partial}{\partial\theta} + b\frac{\partial}{\partial\phi}\right) \tag{2.5.17}$$

$$\sin(\theta)(\alpha b - \beta a) = -\sin(\theta)a \text{ so} \tag{2.5.18}$$

$$\alpha = 0 \text{ and } \beta = 1 \text{ so} \tag{2.5.19}$$

$$\mathcal{X}_H = \frac{\partial}{\partial\phi} \tag{2.5.20}$$

This means the Hamiltonian flow is given by rotation around the vertical axis. Since the energy function is the height function, that energy is conserved by this flow.

Example 2.5.21. An important symplectic manifold to consider is the cotangent bundle to a manifold, where the base manifold is the configuration space of the system. This manifold determines the restrictions on the position coordinates. The cotangent space at each point represents the space of momenta for a particle with that position, which has no restrictions. If $\{q^k\}$ are the local coordinates of the base manifold, the form can be written in the coordinates $\{q^i, p_i\}$ as

$$\omega = d\theta = dp_k \wedge dq^k. \tag{2.5.22}$$

This form is globally exact, so there exists a 1-form θ such that $d\theta = \omega$. This 1-form, called the symplectic potential, is given by

$$\theta = p_k dq^k. \tag{2.5.23}$$

Definition 2.5.24. Suppose f and g are two functions on a symplectic manifold (M, ω) which have Hamiltonian vector fields $\mathcal{X}_f$ and $\mathcal{X}_g$ respectively. The *Poisson* *bracket* of f and g is defined in a coordinate independent manner by

$$\{f, g\} := \omega(\mathcal{X}_f, \mathcal{X}_g) \in C^\infty(M). \tag{2.5.25}$$

The bracket gives smooth functions on the manifold the structure of a Lie algebra.

Remark 2.5.26.

$$\{f, g\} = \omega(\mathcal{X}_f, \mathcal{X}_g) = \iota_{\mathcal{X}_g}\iota_{\mathcal{X}_f}\omega = \iota_{\mathcal{X}_g} df = \mathcal{L}_{\mathcal{X}_g} f, \tag{2.5.27}$$

so a function f is constant along integral curves of X_f by the antisymmetry of the Poisson bracket. We can also see the given a function f and its Hamiltonian vector field, $\mathcal{X}_f$, with the symplectic form locally defined as $dp_k \wedge dq^k$,

$$\iota_{X_f}\omega = \iota_{X_f} dp_k \wedge dq^k = df \tag{2.5.28}$$

is satisfied by

$$X_f = \frac{\partial f}{\partial q^k}\frac{\partial}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial}{\partial q^k}, \tag{2.5.29}$$

so

$$\{f, g\} = \frac{\partial f}{\partial q^k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q^k}. \tag{2.5.30}$$

as asserted in formula (2.1.20).

Definition 2.5.31 (Polarization)**.** If a manifold is equipped with only a symplectic structure, then there is no obvious way of determining which coordinates should play the role of position and which should play the role of momentum. This distinction is obtained by a process called *polarization*. The manifolds we will be concerned with will have additional structure which will help determine which coordinates act as position and which act as momentum. In particular, when we need to consider sections of line bundles, the holomorphic and anti-holomorphic sections will provide this distinction.

Remark 2.5.32. For more information to the ideas of symplectic geometry, see [dS00]. For more on the connections between mechanics and symplectic geometry, see [AM78].

Chapter 3
Geometric Quantization

Motivation 3.0.1. Geometric quantization is a tool for constructing quantum analogues of classical systems. The ultimate goal of quantization is to map observables in a classical setting to observables in a quantum setting. The classical mechanics ideas needed, in the form of symplectic geometry from the Hamiltonian point of view, have already been discussed. In quantum mechanics, observables are given by operators acting on a Hilbert space. Quantizations are not unique, but this should be no surprise. The connection between quantum mechanics and classical mechanics is clearer if we consider taking the limit as $\hbar$ tends to 0 of a quantum system to give a classical analogue. There may are multiple quantum systems which have the same classical limit, and so we should expect ambiguity in trying to find "the" quantum analogue for a given classical system. In order to discuss quantum mechanics mathematically, we will need to introduce a few concepts. For a complete introduction to these ideas, see [Sim68] and [Wei].

Definition 3.0.2. A *wave function* in one space dimension is a complex function of position, ψ, such that $|\psi|^2$ is a probability distribution. The probability of finding a given particle between positions a and b is given by $\int_a^b |\psi(q)|^2 \, dq$.

Definition 3.0.3 (The Schrödinger equation)**.** The time evolution of a wave function is governed by the *Schrödinger equation*. Given a Hamiltonian $\mathcal{H}$ and wave function ψ,

$$\mathcal{H}\psi = -i\hbar\partial_t\psi. \tag{3.0.4}$$

Definition 3.0.5 (Quantization Requirements)**.** The quantization map will be denoted by a hat, $\hat{}$. Dirac determined that the map should be linear and have the following properties:

1. The identity element of the algebra should map to the identity operator in the Hilbert space.

2. Complex conjugation in the Poisson algebra should commute with the mapping. If a star denotes the conjugation of a function and also the adjoint of an operator, this requirement can be written in the symmetric form $\widehat{(F^*)} = (\hat{F})^*$. This suggests that real classical observables will map to Hermitian operators. This is significant because the eigenvalues of a quantum mechanical observable are the possible measurements, and the measurements should be real, as in the case of a Hermitian operator.

3. $\{F, G\}$ should map to $[\hat{F}, \hat{G}]_h := \frac{i}{\hbar}(\hat{F}\hat{G} - \hat{G}\hat{F})$. This requirement gives a time evolution for a quantum mechanical observable. Recall the time evolution for a classical observable f is given by

$$\dot{f} = \{f, H\} \tag{3.0.6}$$

which, after quantization, takes the form

$$\frac{d}{dt}\hat{f} = [\hat{f}, \hat{H}]_h. \tag{3.0.7}$$

4. A *complete set* is defined as a set $F_1, F_2, ..., F_k$ such that $\{f, F_j\} = 0 \;\; \forall j$ implies f is a constant or $[f, F_j]_h = 0 \;\; \forall j$ implies f is a constant. Complete sets should map to complete sets.

In the classical case, the state of a system is given by a linear functional $C^\infty(M) \to \mathbb{R}$, where $1 \to 1$. In quantum mechanics, the set of observables is given by self-adjoint linear operators.

Definition 3.0.8 (Quantization and pre-quantization)**.** It is a result of Groenwald and van Hove that the quantization conditions cannot all be satisfied for a general symplectic manifold, though quantizations do exist for many specific manifolds. A mapping which satisfies the four properties above is called a *quantization*. If only the first three conditions are satisfied, the map is a *pre-quantization*. The details of the two processes are more different than might appear. We will be only concerned with pre-quantization as this is relevant to the dispersionless limit of the Toda lattice. As such, we will refer to "quantization" in later sections to discuss what is more properly "pre-quantization."

To illustrate this distinction, we consider two examples.

Example 3.0.9 (A pre-quantization which is not a quantization)**.** Suppose $M = \mathbb{R}^2 = T^*\mathbb{R}$, and $f \in C^\infty(\mathbb{R})$. We denote the operator of multiplication by x by M_x. Consider the mapping

$$p \to \hat{p} = -\frac{h}{2\pi i}\frac{\partial}{\partial q} \text{ and} \tag{3.0.10}$$

$$q \to \hat{q} = M_q + \frac{h}{2\pi i}\frac{\partial}{\partial p}. \tag{3.0.11}$$

First, $\{q,p\} = 1$, so we require $[\hat{q},\hat{p}]_\hbar = 1$, which is true because:

$$[\hat{q},\hat{p}]f = [M_q + \frac{h}{2\pi i}\frac{\partial}{\partial p}, -\frac{h}{2\pi i}\frac{\partial}{\partial q}]f \tag{3.0.12}$$

$$= \left(-\frac{\hbar}{i}q\frac{\partial}{\partial q} + \hbar^2\frac{\partial}{\partial p}\frac{\partial}{\partial q} + \frac{\hbar}{i}\frac{\partial}{\partial q}M_q - \hbar^2\frac{\partial}{\partial q}\frac{\partial}{\partial p}\right)f \tag{3.0.13}$$

$$= -\frac{\hbar}{i}qf_q + \hbar^2 f_{qp} + \frac{\hbar}{i}\frac{\partial}{\partial q}(qf) - \hbar^2 f_{pq} \tag{3.0.14}$$

$$= \frac{\hbar}{i}(f + qf_q - qf_q) = \frac{\hbar}{i}f \tag{3.0.15}$$

so

$$[\hat{q},\hat{p}]_\hbar f = \frac{i}{\hbar}[\hat{q},\hat{p}]f = \frac{i}{\hbar}\frac{\hbar}{i}f = f. \tag{3.0.16}$$

We will not worry much about the second condition. The problem occurs when we consider general polynomials. We begin by considering quantizations of q and p, $\hat{q}$ and $\hat{p}$. If F is a polynomial, we would like the mapping to extend so that

$F(q,p) \mapsto F(\hat{q},\hat{p})$. Problems arise for general polynomials (which is closely related to the Groenwald and van Hove theorem).

The third condition is a computation. $[q,q]$ and $[p,p]$ are both 0, so there is only one bracket to check, which was done in (3.0.16).

Now we will show that the fourth condition is not satisfied. If $\{f,q\} = \{f,p\} = 0$, then $\frac{\partial f}{\partial p} = \frac{\partial f}{\partial q} = 0$, so f is constant, therefore $\{q,p\}$ form a complete set. (Note this also shows q^i and p_i form a complete set for $\mathbb{R}^{2n}$.) The set of operators $\{\hat{q},\hat{p}\}$ do not form a complete set, however. To see this, we can consider either the operator $\frac{\partial}{\partial p}$ or $\frac{\partial}{\partial q} + \frac{2\pi i}{h}p$. (Either operator will suffice.) Each commutes with $\hat{q}$ and $\hat{p}$, but is not constant. We see here that $\frac{\partial}{\partial p}$ commutes with both $\hat{q}$ and $\hat{p}$:

$$[\frac{\partial}{\partial p},\hat{q}]f = [\frac{\partial}{\partial p}, M_q + \frac{h}{2\pi i}\frac{\partial}{\partial p}]f \tag{3.0.17}$$

$$= \left(\frac{\partial}{\partial p}M_q + \frac{h}{2\pi i}\frac{\partial}{\partial p}\frac{\partial}{\partial p} - M_q\frac{\partial}{\partial p} - \frac{h}{2\pi i}\frac{\partial}{\partial p}\frac{\partial}{\partial p}\right) \tag{3.0.18}$$

$$= \left(\frac{\partial}{\partial p}M_q - M_q\frac{\partial}{\partial p}\right) f = \frac{\partial}{\partial p}qf - qf_p \tag{3.0.19}$$

$$= qf_p - qf_p = 0 \tag{3.0.20}$$

$$[\frac{\partial}{\partial p},\hat{p}]f = [\frac{\partial}{\partial p}, -\frac{\hbar}{i}\frac{\partial}{\partial q}]f \tag{3.0.21}$$

$$= -\frac{\hbar}{i}(\frac{\partial}{\partial p}\frac{\partial}{\partial q} - \frac{\partial}{\partial q}\frac{\partial}{\partial p})f \tag{3.0.22}$$

$$= 0 \tag{3.0.23}$$

Example 3.0.24 (A quantization of $\mathbb{R}^2$)**.** Consider $\mathbb{R}^2$ with coordinates given by (q,p). For the quantization, we will consider $\mathcal{H} = L^2(\mathbb{R}^2)$. The operators will be

$$\hat{q} = M_q \tag{3.0.25}$$

$$\hat{p} = i\hbar\frac{d}{dq}. \tag{3.0.26}$$

$\hat{q}$ is called the position operator and $\hat{p}$ the momentum operator. For thoroughness, we will show how Dirac's third property regarding brackets is satisfied. Since there are only 2 coordinates, we need to check only one bracket since $\{q, q\} = \{p, p\} = 0$ and $\{q, p\} = -\{p, q\}$.

$$\{q, p\} \to [\hat{q}, \hat{p}]_h \tag{3.0.27}$$

so we need to check $[\hat{q}, \hat{p}]_h$:

$$[\hat{q}, \hat{p}]_h = [M_q, i\hbar\frac{d}{dq}] = M_q i\hbar\frac{d}{dq} - i\hbar\frac{d}{dq}M_q = -M_q\frac{d}{dq} + \frac{d}{dq}M_q \tag{3.0.28}$$

which is what we hope since

$$[\hat{q}, \hat{p}]\psi = -q\frac{d}{dq}\psi + \frac{d}{dq}q\psi = -q\psi_q + \psi + q\psi_q = \psi, \tag{3.0.29}$$

so $[\hat{q}, \hat{p}]$ is the identity operator.

If we have a nontrivial potential field V, then the energy operator is given by

$$\hat{H} = \frac{\hbar}{2m}\frac{d^2}{dq^2} + \hat{V}(q). \tag{3.0.30}$$

Remark 3.0.31. The distinction between pre-quantization and quantization is also illustrated by the dimensionality of the Hilbert space. The prequantization maps to $L^2(\{q, p\}, d\mu)$, which are functions of $2n$ variables. The wave function should be a function of position only, so this is not reasonable. The space should be $L^2(q, d\mu)$ instead. This is the case of (3.0.25) and (3.0.26).

Example 3.0.32 (Quantum Mechanical Simple Harmonic Oscillator)**.** We recall the simple harmonic oscillator given earlier, (2.1.14). Now we will scale our units for convenience so that we may consider this hamiltonian to be

$$\mathcal{H} = q^2 + p^2. \tag{3.0.33}$$

Under the mapping described above in (3.0.25) and (3.0.26),

$$\hat{\mathcal{H}} = -\hbar^2\frac{d^2}{dq^2} + M_{q^2}. \tag{3.0.34}$$

Given $\psi(q) \in L^2(\mathbb{R})$,

$$\hat{\mathcal{H}}(\psi(q)) = -\hbar^2\psi''(q) + q^2\psi(q). \tag{3.0.35}$$

Motivation 3.0.36. The choices made for $\hat{q}$ and $\hat{p}$ were determined by physical considerations. In the classical case, possible measurements are given by functions of the q^i and p_i variables. Measurements of quantum mechanical systems are given by eigenvalues of operators. We would like to define an operator corresponding to position, and so we will define it such that the position, q^i, is the eigenvalue of the operator $\hat{q}^i$:

$$\hat{q^i}\psi = q^i\psi. \tag{3.0.37}$$

Notice this is precisely (3.0.25). $\hat{p}$ should be the differentiation operator, though our definition of the bracket will require an additional coefficient:

$$\hat{p}_i = i\hbar\frac{\partial}{\partial q^i}, \tag{3.0.38}$$

exactly as in (3.0.26).

Motivation 3.0.39. To achieve a quantization, we could also use a result of Segal from 1960 which shows that a quantization of an exact symplectic manifold can be given in the following way.

Theorem 3.0.40 (Segal)**.** *Suppose (M, ω) is an exact symplectic manifold with $\omega = d\theta$ and let the vector field $\mathcal{X}_f$ be defined by $\iota_{\mathcal{X}_f}\omega = df$. A quantization is given by*

$$\hat{f} = M_f - i\hbar\mathcal{X}_f - \langle\mathcal{X}_f, \theta\rangle \tag{3.0.41}$$

The proof will be delayed until the ideas which lead to the statement of the theorem are developed.

Lemma 3.0.42.

$$[\mathcal{X}_f, \mathcal{X}_g] = -\mathcal{X}_{\{f,g\}}. \tag{3.0.43}$$

Proof. Lie derivatives will be computationally useful since, for vector fields X and Y, $[X,Y] = \mathcal{L}_X Y$. Since

$$\mathcal{L}_{\mathcal{X}}(\iota_Y \omega) = \iota_{\mathcal{L}_{\mathcal{X}} Y}\omega + \iota_Y \mathcal{L}_{\mathcal{X}}\omega \tag{3.0.44}$$

and using the assumption that $\iota_{\mathcal{X}_H}\omega = dH$,

$$\begin{aligned}
\iota_{[\mathcal{X}_f, \mathcal{X}_g]}\omega &= \iota_{\mathcal{L}_{X_f} X_g}\omega && (3.0.45)\\
&= \mathcal{L}_{\mathcal{X}_f}(\iota_{\mathcal{X}_g}(\omega)) - \iota_{\mathcal{X}_g}(\mathcal{L}_{\mathcal{X}_f}(\omega)) && (3.0.46)\\
&= \mathcal{L}_{\mathcal{X}_f}(dg) - 0 \text{ (by (2.5.9))}\\
&= d\{g,f\} = -d\{f,g\}\\
&= -\iota_{\mathcal{X}_{\{f,g\}}}\omega\\
\implies [\mathcal{X}_f, \mathcal{X}_g] &= -\mathcal{X}_{\{f,g\}}
\end{aligned}$$

□

Remark 3.0.47. The lemma suggests

$$f \to -i\hbar \mathcal{X}_f \tag{3.0.48}$$

may satisfy the quantization conditions, since the third condition is satisfied by the above result. The second is also satisfied, but the first is not as any constant function is assigned to the zero vector field. A possible solution may seem to be adding the operation of multiplication by the function itself:

$$f \to -i\hbar \mathcal{X}_f + M_f. \tag{3.0.49}$$

This mapping satisfies the first property, but the third property is now no longer satisfied. At this point, we will now show that (3.0.41) does satisfy the third of the properties of a quantization in the case of an exact symplectic manifold.

Proof of theorem (3.0.40)**, Segal's Formula.** The first and second requirements for a quantization are trivial. The third is verified as follows. Consider a 1-form θ

which is a symplectic potential on the manifold (M, ω) (so $d\theta = \omega$) and vector fields X and Y. By definition,

$$d\theta(X,Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y]), \text{ so}$$
$$X(\theta(Y)) - Y(\theta(X)) = d\theta(X,Y) + \theta([X,Y]). \tag{3.0.50}$$

Segal's quantization formula is given by

$$\hat{f} = M_f - \hbar i \mathcal{X}_f - \theta(\mathcal{X}_f). \tag{3.0.51}$$

Then,

$$\begin{aligned}
[\hat{f}, \hat{g}] &= [f - \hbar i \mathcal{X}_f - \theta(\mathcal{X}_f), g - \hbar i \mathcal{X}_g - \theta(\mathcal{X}_g)] \\
&= \hbar i \mathcal{X}_g f - \hbar i \mathcal{X}_f g - \hbar^2 \mathcal{X}_{\{f,g\}} + i\hbar d\theta(\mathcal{X}_f, \mathcal{X}_g) - i\hbar\theta([\mathcal{X}_f, \mathcal{X}_g]) \\
&= 2\hbar i\{g, f\} - \hbar^2 \mathcal{X}_{\{f,g\}} + i\hbar d\theta(\mathcal{X}_f, \mathcal{X}_g) - i\hbar\theta(\mathcal{X}_{\{f,g\}}) \\
&= 2\hbar i\{g, f\} - \hbar^2 \mathcal{X}_{\{f,g\}} + i\hbar \omega(\mathcal{X}_f, \mathcal{X}_g) - i\hbar\theta(\mathcal{X}_{\{f,g\}}) \\
&= -\hbar i\{f, g\} - \hbar^2 \mathcal{X}_{\{f,g\}} - i\hbar\theta(\mathcal{X}_{\{f,g\}}) \\
&= i\hbar(\{f, g\} + i\hbar \mathcal{X}_{\{f,g\}} - \theta(\mathcal{X}_{\{f,g\}}) \\
&= i\hbar\widehat{\{f, g\}}
\end{aligned} \tag{3.0.52}$$

□

Remark 3.0.53. The phase spaces considered so far are cotangent bundles to manifolds. While an important source of examples, they do not exhaust all possibilities. If we consider a phase space given by the sphere, $x_1^2 + x_2^2 + x_3^2 = r^2$ and take ω to be the usual area form, local coordinates are given by $u = x_1, v = x_2$ and the form by $\omega = \dfrac{r(du \wedge dv)}{\sqrt{r^2 - u^2 - v^2}}$. This manifold cannot be represented as a cotangent bundle. There exist complex polarizations for this manifold, which comes from a condition on the surface area of the manifold, which will be described in generality later.

Motivation 3.0.54. By Segal's result, Theorem (3.0.40), we were able to construct a formula for quantization of symplectic manifold, given that the form was exact. Since we want to quantize arbitrary compact manifolds, we will need additional structure. We know that the form is always exact locally, so we can cover M by open sets U_α such that on each open set there is a 1-form θ_α where $d\theta_\alpha = \omega$. We could then use Segal's formula on each set to achieve a quantization there. If the manifold admits a quantization, and not all manifolds do, then these local operators can be glued together to form a global operator. The problem with this construction is that the operators do not act on functions of the manifold, but instead on sections of a line bundle over the manifold. We will need to quickly discuss line bundles and connections before we progress.

Remark 3.0.55. The ideas of geometric quantization can be explored further through [EEMLRRVM98], [Ger],[Kos70], [Rit02], and [Woo92].

Chapter 4
Line Bundles

Motivation 4.0.1. We might hope that complex holomorphic functions on a phase space M are the primary objects to deal with. The only holomorphic functions on compact connected Kähler manifolds, however, are just constants, so we will need something more complicated. Not all manifolds require that we use line bundles. Consider the special case of the phase space $\mathbb{R}^2$. The line bundles we will need are trivial, and lead to ordinary holomorphic functions.

Definition 4.0.2. A *line bundle*, L over a manifold M is a smooth manifold equipped with a smooth surjection $\pi : L \to M$ such that:

1. the fibre $\pi^{-1}(m) = L_m \cong \mathbb{C}$ for all $m \in M$, and
2. (local triviality) for every $m \in M$, there exists an open neighborhood U_m and a diffeomorphism $\varphi : \pi^{-1}(U_m) \to U_m \times \mathbb{C}$ so that $\phi(L_m) \subset \{m\} \times \mathbb{C}$ and $\varphi|_{L_m}$ is a linear isomorphism.

Example 4.0.3. The simplest example of a line bundle over a manifold, M is $M \times \mathbb{C}$. The vector space at each point m is $\{m\} \times \mathbb{C} \cong \mathbb{C}$.

Definition 4.0.4. A *section* of a line bundle is a map $s : M \to L$ such that $\pi \circ s = id_M$.

Definition 4.0.5. A line bundle is called a *trivial line bundle* if there exists a global diffeomorphism $\phi : L \to M \times \mathbb{C}$.

Proposition 4.0.6. *A line bundle is trivial if and only if it has a nowhere vanishing section.*

Proof. (⇒) Suppose the line bundle L is trivial. This means we can use the entire bundle L as the domain of φ so that $\varphi : L \to M \times \mathbb{C}$ is the trivialization. $\varphi^{-1}(m, 1)$ gives a nowhere vanishing section on L.
(⇐) If L has a nowhere vanishing section, $s : M \to L$, then a trivialization is given by $(m, \lambda) \to \lambda s(m)$. □

Motivation 4.0.7. Line bundles can be studied by investigating the transition functions between charts. Consider a line bundle L over M. By local triviality, we can cover M by open sets $\{U_\alpha\}$ so that each U_α has a nonvanishing section $s_\alpha : U_\alpha \to L$ (i.e. $s_\alpha(U_\alpha)$ is a trivial line bundle by the proposition). Now consider a global section S (possibly vanishing). One can restrict the global section S to the open sets $\{U_\alpha\}$. This will allow us to compare the global section to the nonvanishing local sections. There exist functions $f_\alpha : U_\alpha \to \mathbb{C}$ such that

$$f_\alpha s_\alpha = S|_{U_\alpha} \tag{4.0.8}$$

is defined naturally. (Notice the fact that s_α is nonvanishing is essential here.) In addition, we can consider the overlaps between the charts. Suppose $U_\alpha \cap U_\beta \neq 0$. Then,

$$S|_{U_\alpha \cap U_\beta} = f_\alpha s_\alpha|_{U_\alpha \cap U_\beta} = f_\beta s_\beta|_{U_\alpha \cap U_\beta}. \tag{4.0.9}$$

Alternatively, we can consider a set of nonvanishing sections $\{s_\alpha\}$. If there exist $\{f_\alpha : U_\alpha \to \mathbb{C}\}$ satisfying $f_\alpha s_\alpha = f_\beta s_\beta$ for all α and β where $U_\alpha \cap U_\beta \neq 0$, then the set $\{s_\alpha\}$ form a global section, S.

Definition 4.0.10. The *transition functions* for a line bundle L over M with local sections $\{s_\alpha\}$ over open sets $\{U_\alpha\}$ are functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}, \tag{4.0.11}$$

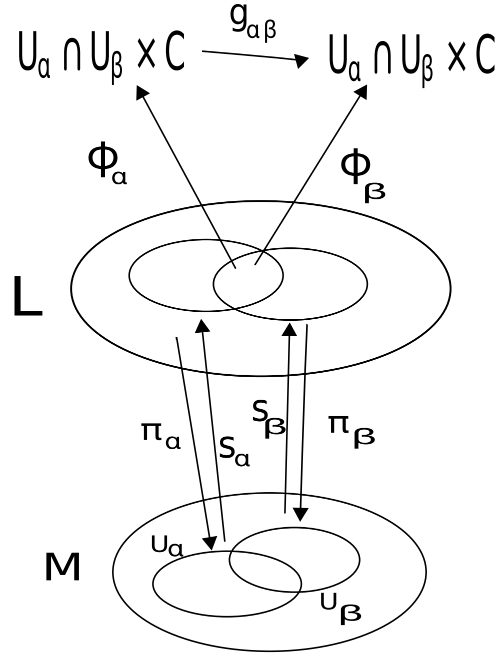
where, for $z \in \mathbb{C}$, $g_{\alpha\beta}$ is defined by

$$\phi_\alpha \phi_\beta^{-1}(m, z) \to (m, g_{\alpha\beta}). \tag{4.0.12}$$

This takes z in the β chart to $g_{\alpha\beta}(z)$ in the α chart, so for sections s_α and s_β,

$$s_\alpha = g_{\alpha\beta} s_\beta. \tag{4.0.13}$$

The relations may be best seen in a diagram:



Proposition 4.0.14. *These transition functions must satisfy the following conditions:*

1. $g_{\alpha\alpha} = 1$
2. $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ *on* $U_\alpha \cap U_\beta$
3. $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$ *on* $U_\alpha \cap U_\beta \cap U_\gamma$

Remark 4.0.15. These three properties will provide reflexivity, symmetry, and transitivity for an equivalence relation on the disjoint union $\mathbb{C} \sqcup U_\alpha$. The third property is called the *cocycle condition.*

Proof. 1. $s_\alpha = g_{\alpha\beta} s_\beta$, so

$$s_\alpha = g_{\alpha\alpha} s_\alpha \Rightarrow g_{\alpha\alpha} = 1.$$

2. On $U_\alpha \cap U_\beta$,

$$\begin{aligned} s_\alpha &= g_{\alpha\beta} s_\beta \\ &= g_{\alpha\beta} g_{\beta\alpha} s_\alpha \\ \Rightarrow g_{\alpha\beta} g_{\beta\alpha} &= 1 \\ g_{\beta\alpha} &= g_{\alpha\beta}^{-1}. \end{aligned}$$

3. On $U_\alpha \cap U_\beta \cap U_\gamma$,

$$\begin{aligned} s_\alpha &= g_{\alpha\beta} s_\beta \\ &= g_{\alpha\beta} g_{\beta\gamma} s_\gamma \text{ but} \\ s_\alpha &= g_{\alpha\gamma} s_\gamma, \text{ so} \\ g_{\alpha\beta} g_{\beta\gamma} &= g_{\alpha\gamma} \text{ thus} \\ g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} &= 1 \end{aligned}$$

□

Proposition 4.0.16. *Given a manifold M, an open cover $\{U_\alpha\}$, and functions $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}\}$ satisfying the three conditions of proposition (4.0.14) for every U_α and U_β, there exists a line bundle $L \to M$ having $\{g_{\alpha\beta}\}$ as transition functions.*

Proof. Consider the disjoint union $\{\bigsqcup_\alpha U_\alpha \times \mathbb{C}\}$. Consider an element of this disjoint union, (m, λ). If I is the indexing set for the open sets U_α, consider $I \times \bigsqcup_\alpha U_\alpha \times \mathbb{C}$ with elements (α, m, λ), so that $\alpha \in I, m \in U_\alpha \subset M, \lambda \in \mathbb{C}$. Now impose the relation $\sim$ where $(\alpha, m, \lambda) \sim (\beta, n, \mu)$ if $m = n$ and $g_{\alpha\beta}(\mu) = \lambda$. The relation $\sim$ is an equivalence relation because the three conditions of proposition (4.0.14) are precisely those of reflexivity, symmetry, and transitivity. Addition and scalar multiplication are defined for equivalence classes in the third coordinate. If brackets denote equivalence classes, define the section map by

$$s_\alpha(m) = [(\alpha, m, 1)]. \tag{4.0.17}$$

The result now follows since

$$\begin{aligned} g_{\alpha\beta}s_\beta(m) &= g_{\alpha\beta}[(\beta, m, 1)] \\ &= [(\beta, m, g_{\alpha\beta}1)] \\ &= [(\alpha, m, 1)] = s_\alpha(m). \end{aligned}$$

The last detail needed is to show that the resulting space is actually a line bundle. The mapping $\phi_\alpha : [(\alpha, m, \lambda)] \mapsto (m, \lambda)$ gives a local trivialization. The projection map is given by $\pi : [(\alpha, m, \lambda)] \mapsto m$. □

Remark 4.0.18. We will denote the set of sections of a line bundle L over a manifold M by $\Gamma(L, M)$, which then has the structure of a C^∞-module with multiplication given by

$$(fs)(m) = f(m)s(m)\ \forall f \in C^\infty(M),\ s \in \Gamma(L, M). \tag{4.0.19}$$

Suppose $U \subset M$ is an open set which admits a non-vanishing section. We can identify smooth functions on U with section of the line bundle over U. Suppose $f \in C^\infty(U)$ and s is such a nonvanishing section on $U \subset M$.

$$s(U) = \phi^{-1}(U \times \mathbb{C}) \tag{4.0.20}$$

This means that for every $m \in U$,

$$s(m) = \phi^{-1}(m, f(m)). \tag{4.0.21}$$

Definition 4.0.22. Line bundles have a *tensor product structure* defined fibre-wise. Suppose L_1 and L_2 are line bundles over M. $L_1 \otimes L_2$ denotes the tensor product of the two line bundles. Elements of this tensor product are equivalence classes such that $[a, b] = [\hat{a}, \hat{b}]$ if $\exists c \in \mathbb{C}$ such that $[\frac{1}{c}a, cb] = [\hat{a}, \hat{b}]$.

Properties 4.0.23. By choosing appropriate subcovers for each of L_1 and L_2, we may assume that the open sets defining the line bundles are identical. Suppose

$\pi_1 : L_1 \to M$ and $\pi_2 : L_2 \to M$. $\pi : L_1 \otimes L_2 \to M$, which is defined coordinate-wise. The implies that the tensor product of complex line bundles is again a complex line bundle.

Chapter 5
Connections and Curvature

5.1 Connections

Motivation 5.1.1. One motivation for the concept of a connection comes from attempting to differentiate sections of line bundles. In differentiating, one will be forced to compute the difference of vectors which are located in distinct vector spaces. The connection allows one to reconcile this distinction. The connection should measure how a section is changing in the direction of a specific vector field, X. We will denote the line bundle over M by L, as in the previous section.

Definition 5.1.2. A *connection* is a $\mathbb{K}$-linear (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) map $\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$ such that $\forall f \in C^\infty(M), u \in C^\infty(E), \nabla(fu) = df \otimes u + f\nabla u$.

Remark 5.1.3. The connection along a vector field X is a map $\nabla_X : \Gamma(L) \to \Gamma(L)$.

Properties 5.1.4. The definition of the connection implies the following properties:

1. The map to respect scaling by functions on the manifold, so we will require

$$\nabla_{fX} u = f \nabla_X u \;\; \forall f \in C^\infty(M). \tag{5.1.5}$$

2. The connection ∇ should satisfy a Leibniz (product) rule,

$$\nabla_X(fu) = (Xf)u + f\nabla_X u \forall f \in C^\infty(M), u \in C^\infty(E). \tag{5.1.6}$$

3. The connection should satisfy

$$\nabla_{X+Y}\ \phi = \nabla_X\ \phi + \nabla_Y\ \phi \tag{5.1.7}$$

Motivation 5.1.8. The connection can be defined in terms of coordinates using functions called the Christoffel symbols. We can begin to find a derivative of a vector field by first noticing the special case

$$\nabla_i \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}. \tag{5.1.9}$$

Definition 5.1.10. The functions Γ^k_{ij} are called the *Christoffel symbols*.

Properties 5.1.11. A general vector field X can be written as $X^j \frac{\partial}{\partial x^j}$. By the Leibniz rule,

$$\nabla_i(X) = \nabla_i \left(X^j \frac{\partial}{\partial x^j} \right) \tag{5.1.12}$$

$$= \frac{\partial X^j}{\partial x^i} \frac{\partial}{\partial x^j} + X^j \Gamma^k_{ij} \frac{\partial}{\partial x^k} \tag{5.1.13}$$

$$= \left(\frac{\partial X^k}{\partial x^i} + X^j \Gamma^k_{ij} \right) \frac{\partial}{\partial x^k} \tag{5.1.14}$$

The connection can now be extended to a vector field $Y = Y^i \frac{\partial}{\partial x^i}$ by

$$\nabla_Y X = \nabla_{Y^i \frac{\partial}{\partial x^i}} X = Y^i \left(\nabla_i X \right) \tag{5.1.15}$$

If the manifold M is also equipped with a metric, h, then it is natural to require that the two structures interact in a specified way.

Definition 5.1.16. A connection is said to be *compatible with the metric h* if

$$\nabla_X (h(Y, Z)) = h(\nabla_X Y, Z) + h(Y, \nabla_X Z) \tag{5.1.17}$$

Proposition 5.1.18. *Every line bundle admits a connection.*

Sketch of proof. Consider a line bundle $\pi : L \to M$ which has an open cover $\{U_\alpha\}$ with nowhere vanishing sections $\{s_\alpha\}$. Let a partition of unity subordinate to the choice U_α be denoted by p_α, and let S be a global section. S may vanish, so we would like to compare S with nonvanishing sections. As before, define functions $\{f_\alpha\}$ by the condition $S|_{U_\alpha} = f_\alpha s_\alpha$. We now claim a connection is given by

$$\nabla(S) = \sum df_\alpha p_\alpha s_\alpha. \tag{5.1.19}$$

It must be checked that the above equation actually defines a connection. Consider $g \in C^\infty(M)$. $\nabla(gS)$ will need to be defined by considering the local sections. Now, using the Leibniz rule for d,

$$\begin{aligned} \nabla(gS) &= \sum d(g|_{U_\alpha} f_\alpha p_\alpha s_\alpha) \\ &= \sum (dg) f_\alpha p_\alpha s_\alpha + g \sum df_\alpha p_\alpha s_\alpha \\ &= dg \sum f_\alpha p_\alpha s_\alpha + g\nabla(S) = dg(S) + g\nabla(S). \end{aligned} \tag{5.1.20}$$

□

Now suppose $\hat{s}_1(x)$ and $\hat{s}_2(x)$ are local representations for sections $s_1(x)$ and $s_2(x)$ where $x \in M$. Also suppose $\hat{h}(x)$ is an $\mathbb{R}$-valued function on M. Consider the Hermitian metric on sections of a line bundle given by

$$h(s_1, s_2)(x) = \hat{h}(x)\bar{\hat{s}}_1(x)\hat{s}_2(x). \tag{5.1.21}$$

A natural Hermitian metric to keep in mind is one on the trivial bundle:

$$h(m, z_1), (m, z_2) = z_1\bar{z}_2 \ \forall (m, z_i) \in (M, \mathbb{C}). \tag{5.1.22}$$

Definition 5.1.23. Suppose ∇ is a connection on a line bundle L over M. Consider a local trivialization $\phi_i : U_i \to \mathbb{C} \times M$. The connection acts on open sets by determining a 1-form called the *potential 1-form*. On a given open set U with connection ∇, the connection acts on a section s by the rule

$$\nabla s = -i\theta s. \tag{5.1.24}$$

Because it will arise often, we denote

$$\eta = -i\theta. \tag{5.1.25}$$

Proposition 5.1.26. *The connection ∇ can be globally defined.*

Proof. By setting

$$\nabla = d + \eta, \tag{5.1.27}$$

we will show that the connection transforms appropriately under a change of chart. Consider a connection ∇ with a trivializing section s_0 on U_0 so that

$$\nabla s_0 = \eta s_0. \tag{5.1.28}$$

If ϕ is a (complex-valued) transition function to another set U_1, the trivial section in U_1 is of the form $s_1 = \phi s_0$ on $U_0 \cap U_1$, so

$$\nabla s_1 = \nabla(\phi s_0) = (d\phi)s_0 + \phi(\nabla s_0) \tag{5.1.29}$$

$$= (d\phi)s_0 + \phi(\eta s_0) \tag{5.1.30}$$

$$= (d\phi + \eta\phi)s_0 \tag{5.1.31}$$

$$= (d + \eta)\phi s_0 = (d + \eta)s_1 \tag{5.1.32}$$

$$\Rightarrow \nabla = d + \eta. \tag{5.1.33}$$

□

5.2 Curvature

Definition 5.2.1. The *curvature 2-form* of the connection ∇ with potential 1-form θ is given by $d\theta$. The curvature $d\theta$ will be denoted Ω.

Motivation 5.2.2. Recall that the potential 1-forms, defined in 5.1.23, were defined on the charts. Though these forms depend on the chart, we claim that the curvature $d\theta$ is independent of the chart.

Proposition 5.2.3. *The curvature, Ω, is globally defined.*

Proof. The idea of this proof will be to show that the difference between the locally defined θ's is an exact form, so the curvature does not change as the local chart changes. Recall (5.1.6), which was the Leibniz rule for the connection. Consider two charts U_0 and U_1 with trivializing sections s_0 and s_1 respectively. Suppose that on U_0, $\nabla(s_0) = \eta s_0$ and on U_1, $\nabla_X(s_1) = \hat{\eta} s_1$. The transition between the sections is given by $s_1 = \phi s_0$ on $U_0 \cap U_1$. On this intersection,

$$\nabla(s_1) = \hat{\eta} s_1 = \hat{\eta}\phi s_0. \tag{5.2.4}$$

At a vector field X,

$$\nabla_X(s_1) = \hat{\eta}\phi s_0(X). \tag{5.2.5}$$

Also,

$$\nabla_X(s_1) = \nabla_X(\phi s_0) = (X\phi)s_0 + \phi\nabla_X s_0 = d\phi(X)s_0 + \phi\eta s_0(X) \tag{5.2.6}$$

Therefore,

$$\hat{\eta}\phi s_0(X) = d\phi(X)s_0 + \phi\eta s_0(X) \tag{5.2.7}$$

$$\Rightarrow \hat{\eta}\phi(X) = d\phi(X)s_0 + \phi\eta(X) \tag{5.2.8}$$

$$\Rightarrow \hat{\eta}(X) = \frac{d\phi(X)}{\phi} + \eta(X) \tag{5.2.9}$$

$$\Rightarrow (\hat{\eta} - \eta)(X) = \frac{d\phi(X)}{\phi} = d\ln(\phi)(X) \tag{5.2.10}$$

In other words, changing the chart corresponds to mapping the potential 1-forms by the rule

$$\eta \mapsto \eta + d\ln\phi. \tag{5.2.11}$$

Since switching between charts only differs by an exact form, the curvature $\Omega = d\theta$ is globally defined. □

Remark 5.2.12. We have a complex line bundle L with Hermitian structure h and compatible connection ∇. We can choose the trivialization to satisfy $h(s_j, s_j) = 1\ \forall j$ by simply rescaling.

Proposition 5.2.13. *The one-form η is purely imaginary.*

Proof. We need to consider the compatibility condition for the metric and connection. For a vector field X,

$$dh(X(s,t)) = Xh(s,t) \tag{5.2.14}$$

$$= h(\nabla_X s, t) + h(s, \nabla_X t) \tag{5.2.15}$$

$$= h(\eta(X)s, t) + h(s, \eta(X)t) \tag{5.2.16}$$

$$= \eta(X)h(s,t) + \bar{\eta}(X)h(s,t). \tag{5.2.17}$$

Thus,

$$dh(s,t) = (\eta + \bar{\eta})h(s,t). \tag{5.2.18}$$

In particular,

$$dh(s_0, s_0) = (\eta + \bar{\eta})h(s_0, s_0) = (\eta + \bar{\eta})h_0. \tag{5.2.19}$$

With the assumption that $h(s_0, s_0) = 1$ on the chart, $\eta + \bar{\eta} = 0$, which forces η be imaginary. □

Chapter 6

Cohomology and the Quantization Condition

Motivation 6.0.1. Recall Segal's theorem (3.0.40) allows a quantization to be constructed on a single chart of a manifold. We would like to know when this type of construction can be extended to a global result. This will require a relation between the local structure and the global structure of the manifold. *Cohomology* does this, and we will be particularly interested in the *Cech cohomology*. We begin with a smooth manifold, M.

6.1 Cech Cohomology

Definition 6.1.1. An open cover $\{U_\alpha\}$ of M is called a *contractible cover* of M if each of the open sets $U_i, U_i \cap U_j, U_i \cap U_j \cap U_k \ldots$ is either empty or can be smoothly contracted to a point.

Definition 6.1.2. A *k-simplex* is a $k+1$-tuple of indices $(i_0, i_1, \ldots, i_k)$ determining sets $(U_{i_0}, U_{i_2}, \ldots, U_{i_k})$ so that $U_{i_0} \cap U_{i_2} \cap \ldots \cap U_{i_k} \neq 0$.

Definition 6.1.3. A *k-cochain* is a totally skew (skew in any pair of coordinates) map

$$g : (i_0, i_1, \ldots, i_k) \mapsto g(i_0, i_1, \ldots, i_k) \in \mathbb{R}. \tag{6.1.4}$$

The set of all k-cochains will be denoted $C^k(U, \mathbb{R})$.

Remark 6.1.5. The target space of a k-cochain is generally defined to be an abelian Lie group. For the purposes of this section, we will only need this group to be $\mathbb{R}$. We will also only need the case $k = 2$ for our construction.

Definition 6.1.6. For each k, there is a map $\delta : C^{k+1}(U, \mathbb{R}) \to C^k(U, \mathbb{R})$ defined, using ˆ to denote omission, by

$$\delta g(i_0, i_1, \ldots, i_{k+1}) = \sum_{j=0}^{k+1} (-1)^j g(i_0, i_1, \ldots, \hat{i}_j, \ldots, i_{k+1}) \tag{6.1.7}$$

The operator δ is called the *coboundary operator*.

Fact 6.1.8. $\delta^2 : C^{k+2}(U, \mathbb{R}) \to C^k(U, \mathbb{R})$ is trivial.

Definition 6.1.9. If $g \in C^k(U, \mathbb{R})$ and $\delta g = 0$, then g is called a *k-cocycle*.

Definition 6.1.10. If g is a k-cocycle, and $g = \delta h$ for some $(k+1)$-cochain, h, then g is called a *k-coboundary*. The set of coboundaries is denoted $Z^k(U, \mathbb{R})$.

Definition 6.1.11. The k^{th} *cohomology group* relative to $\{U\}$ is defined as the quotient

$$H^k(U, \mathbb{R}) = Z^k(U, \mathbb{R})/\delta(C^{k-1}(U, \mathbb{R})). \tag{6.1.12}$$

The elements of this group consist of equivalence classes of cocycles with the relation that two cocycles are in the same class if they differ by a coboundary.

Fact 6.1.13. The Cech cohomology of a manifold is independent of the choice of contractible cover. Because of this, we may denote the cohomology by $H^k(M, \mathbb{R})$.

Definition 6.1.14. Let M be triangulated with vertices given by $\{x_i\}$. The *star neighborhood* of the vertex x_j is given by

$$U_j = \{x \in M | \text{x is in the interior of a simplex with a vertex } x_j\}, \tag{6.1.15}$$

where the simplices are given by the triangulation.

Remark 6.1.16. The cohomology groups discussed here are isomorphic with the de Rham cohomology groups. This can be seen clearly in the special case of the star neighborhoods. Let $x_{i_0}, x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ designate the simplex with vertices $x_{i_0}, x_{i_1}, x_{i_2}, \ldots,$

and x_{i_k}. If ψ is a complex k-form on that simplex, then we can produce a k-cochain $\hat{\psi}$ by

$$\hat{\psi}(i_0, i_1, \ldots, i_k) = \int_{x_{i_1}, x_{i_2}, \ldots, x_{i_k}} \psi. \tag{6.1.17}$$

By Stokes' theorem,

$$(d\hat{\psi})(i_0, i_1, \ldots, i_{k+1}) = \delta\psi(i_0, i_1, \ldots, i_{k+1}). \tag{6.1.18}$$

The isomorphism can be constructed from this association of k-forms with k-cochains. The correspondence exists for any contractible cover, not just the star neighborhoods.

6.2 The Quantization Condition

Definition 6.2.1 (The Integrality Condition)**.** A form ω is called *integral* if the class of $(2\pi\hbar)^{-1}\omega$ lies in the image of $H^2(M, \mathbb{Z})$.

Definition 6.2.2. A symplectic manifold (M, ω) is called *quantizable* if ω satisfies the integrality condition above, i.e. ω is integral.

This is equivalent to:

Proposition 6.2.3. *A symplectic manifold is* quantizable *if there exists a Hermitian line bundle $B \to M$ with a connection ∇ on B which has curvature $\hbar^{-1}\omega$.*

Remark 6.2.4. The first definition requires the integral of ω over any closed 2-dimensional surface in M be an integer multiple of $2\pi\hbar$. The B in the second definition is called the *quantizing line bundle*.

Proof that the quantization conditions are equivalent. For both directions, we will suppose $\pi : L \to M$ is a line bundle with local trivialization $\{U_i, s_i\}$. By possibly using a finer cover, we may assume that $\{U_i\}$ is contractible. We will use the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}$ to produce cochains. The transition functions are actually cocycles since $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$. (This is why the condition described in the line bundles section was called the cocycle condition.) By proposition 4.0.16, we can construct a line bundle with transition functions given by the cochains. We now have a link between transition functions in a line bundle and Cech cohomology cochains.

Assume 6.2.3 is true. Therefore, we suppose L is a Hermitian line bundle with connection ∇ having curvature $\hbar^{-1}\omega$. We choose a triangulation, following the idea of a star neighborhood, which has vertices $\{x_i\}$. $\{U_i\}$ will denote the star neighborhoods of the $\{x_i\}$. The connection on a chart U_i is determined by a potential 1-form, α_i. As we computed in the proof of 5.2.3, the potential 1-forms transform on intersections $U_i \cap U_j$ by the rule

$$\alpha_i = \alpha_j + d\ln(\phi) \tag{6.2.5}$$

where ϕ transforms sections on U_j to sections on U_i. Let

$$i\hbar\, df_{ij} = \alpha_i - \alpha_j = d\ln(\phi) \tag{6.2.6}$$

where

$$f_{ij} : U_i \cap U_j \to \mathbb{C} \tag{6.2.7}$$

so that

$$f_{ij} = (2\pi\hbar)\ln(\phi). \tag{6.2.8}$$

Now the cocycle condition is equivalent to $d(f_{ij} + f_{jk} + f_{ki}) = 0$. We can consider

$$a(i,j,k) = \frac{1}{2\pi\hbar}(f_{ij} + f_{jk} + f_{ki}) : U_i \cap U_j \cap U_k \to \mathbb{C}. \tag{6.2.9}$$

This function is a constant since $da = 0$. Since it is constant on each chart and is constant on the overlaps, a is a constant function on the entire manifold (since we are working on a connected manifold). This constant, as it turns out, is always an integer (see [SW76]). The second cohomology class is defined over each 2-simplex Δ_{ijk} which has vertices $x_i, x_j,$ and, x_k and is evaluated by

$$\omega(i,j,k) = \int_{\Delta_{ijk}} \omega. \tag{6.2.10}$$

Next we can evaluate and group so that

$$\omega(i,j,k) = \int_{\Delta_{ijk}} \omega \tag{6.2.11}$$

$$= \frac{1}{3}\int_{\partial\Delta_{ijk}} (\alpha_i + \alpha_j + \alpha_k) \tag{6.2.12}$$

$$= \frac{1}{6}\left[2(f_{ij} + f_{jk} + f_{ki})(x_i) + 2(f_{ij} + f_{jk} + f_{ki}(x_j) + 2(f_{ij} + f_{jk} + f_{ki}(x_k)\right] \tag{6.2.13}$$

$$- \frac{1}{2}\left[(f_{ij}(x_i) + f_{ij}(x_j) + f_{jk}(x_j) + f_{jk}(x_k)) + f_{jk}(x_k) + f_{ki}(x_k) + f_{ki}(x_i)\right] \tag{6.2.14}$$

$$- \frac{1}{2}\int_{x_i x_j} (\alpha_i + \alpha_j) + \int_{x_j x_k} (\alpha_j + \alpha_k) + \int_{x_k x_i} (\alpha_k + \alpha_i) \tag{6.2.15}$$

by using Stokes' theorem. The last two lines in the computation are coboundaries, and so will be quotiented out in the cohomology. The line before that is just the cocycle $2\pi\hbar a(i,j,k)$. Therefore, $\omega(i,j,k) = (2\pi\hbar)a(i,j,k)$, so $\frac{1}{2\pi\hbar}\int_{\partial\Delta_{ijk}} \omega$ is an integer. This tells us that the class of $(2\pi\hbar)^{-1}\omega$ is in $H^2(M,\mathbb{Z})$.

Now we suppose 6.2.2, so that the class of $(2\pi\hbar)^{-1}\omega$ is in $H^2(M,\mathbb{Z})$. We know ω is a closed 2-form, but we will assume that it is not globally exact. On a star neighborhood (defined at the beginning of the proof), $\omega = d\alpha_i$ for some 1-form. This relies on the fact that the neighborhood U_i be contractible, which was why we needed the option of taking a subcover of an original cover. As above, we can define functions f_{ij} so that $\alpha_i - \alpha_j = df_{ij}$. By the same reasoning as above, $a(i,j,k) = f_{ij} + f_{jk} + f_{ki}$ must be constant since $da(i,j,k) = 0$. We need to construct a connection ∇ which has curvature $\hbar^{-1}\omega$. We can do this if we can construct transition functions, which will be based on the $\alpha_i's$. While we don't know if $a(i,j,k)$ is an integer, we do know that it is in a cohomology class with a some $b(i,j,k)$, which is an integer. Let this integer be denoted K. If $a(i,j,k) = f_{ij} + f_{jk} + f_{ki}$, then b must differ by a by an exact form since they are in the same cohomology class. Suppose these differ by dh. This means we can write $\hat{f}_{ij} = f_{ij} + h_{ij}$ for h_{ij} exact, so that $g_{ij} = e^{2\pi i \hat{f}_{ij}}$ defines a cocycle since

$$g_{ij}g_{jk}g_{ki} = e^{2\pi i(\hat{f}_{ij}+\hat{f}_{jk}+\hat{f}_{ki})} = e^{2\pi i K} = 1. \tag{6.2.16}$$

The $g'_{ij}s$ form transition functions, and one can then see that the curvature of this connection is, in fact, ω. For a much more involved and complete exploration of these ideas, the reader is directed to [SW76]. □

Example 6.2.17 (Quantization condition on $\mathbb{S}^2_r$)**.** Any surface can be given the structure of a symplectic manifold by letting ω be a volume form. If this surface is compact, the form cannot be globally exact by Stokes' theorem. Given a compact manifold M without boundary, if $\omega = d\theta$,

$$\mathrm{Vol}_\omega(M) = \int_M \omega = \int_M d\theta = \int_{\partial M} \theta = \int_\emptyset \theta = 0. \tag{6.2.18}$$

The quantization condition imposes specific restrictions on either the radius of the sphere or the symplectic form. Consider the 2-sphere with radius 1, $\mathbb{S}^2$. Using the volume form

$$\omega = k\sin(\theta) d\theta \wedge d\phi, \tag{6.2.19}$$

then ω is only integral if

$$\int_M \omega = 4\pi k = n2\pi\hbar \tag{6.2.20}$$

for $n \in \mathbb{Z}$. This requires $k = \frac{n\hbar}{2}$. This puts discrete conditions on the form. The form could also have been fixed and the radius of the sphere would have a discrete condition. We can also see here that the Poisson bracket for the sphere of radius r must be

$$\{f, g\} = \frac{1}{r^2\sin(\theta)}\left(\frac{\partial f}{\partial \phi}\frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \phi}\right). \tag{6.2.21}$$

We will see later how this will be useful after stereographic projection in the section on quantization of $\mathbb{S}^2$. It would be misleading to suggest that merely scaling the symplectic form will make a symplectic manifold quantizable. One can consider a Cartesian product of 2-spheres with incommensurable radii to see that the process is not so simple.

Chapter 7
Quantization of Kähler Manifolds

Definition 7.0.1. A *complex manifold*, M, is a smooth manifold which has coordinate patches diffeomorphic with $\mathbb{C}^n$ for some fixed n which has holomorphic transition functions.

Definition 7.0.2. A *Hermitian metric* is an assignment of a self-adjoint (Hermitian) form to each fibre of a line bundle.

Definition 7.0.3. Suppose M is a complex manifold which has a smoothly varying Hermitian metric h on its tangent spaces. If the imaginary part of h is closed, M is called a *Kähler manifold*.

Remark 7.0.4. The Hermitian metric of a Kähler manifold gives a method for relating multiple structures on the manifold. Assume M is a complex manifold with a smoothly varying Hermitian metric. The imaginary part of h is a non-degenerate, skew-symmetric 2-form. If M is Kähler, then that 2-form is also closed, so it is a symplectic form, making M into a symplectic manifold. The real part of h assigns a symmetric 2 form to each of the tangent spaces, which gives M the structure of a Riemannian manifold.

Definition 7.0.5. We introduce the following notation

$$\partial = \sum_i dz_i \frac{\partial}{\partial z_i} \tag{7.0.6}$$

$$\bar{\partial} = \sum_i d\bar{z}_i \frac{\partial}{\partial \bar{z}_i} \tag{7.0.7}$$

Remark 7.0.8. This notation allows the immediate decomposition of d as $\partial + \bar{\partial}$. The complex structure of the Kähler manifold gives a natural way of imposing a polarization on the sections. By considering only holomorphic sections, (5.2.19) is written

$$\partial h_0 = h_0 \eta. \tag{7.0.9}$$

Equivalently,

$$\eta = \partial \ln(h_0) \tag{7.0.10}$$

Recalling that $\text{curv}(L) = \Omega = d\eta$, we now have the condition

$$\text{curv}(L) = -\partial\bar{\partial} \ln(h_0) = \bar{\partial}\partial \ln(h_0). \tag{7.0.11}$$

Example 7.0.12 ($\mathbb{C}^n$)**.** If $\mathbb{C}^n$ is endowed with the natural symplectic form $\omega = i \sum_j dz_j \wedge d\bar{z}_j$, the line bundle is trivial and we can find the appropriate Hilbert spaces for the quantization. Since $i \cdot \text{curv}(L) = \omega$, $\omega = -i\partial\bar{\partial} \ln(h_0)$. Therefore,

$$\partial\bar{\partial} \ln(h_0) = -\omega = -\sum_j dz \wedge d\bar{z}. \tag{7.0.13}$$

However, $\partial\bar{\partial} z\bar{z} = \sum_j dz \wedge d\bar{z}$, so

$$\ln(h_0(z)) = -z\bar{z}. \tag{7.0.14}$$

Therefore,

$$h_0(z) = e^{-z\bar{z}} = e^{-|z|^2}. \tag{7.0.15}$$

We then need to consider the particular Hilbert spaces. We can consider arbitrary powers of a line bundle over a manifold, so we could in fact have $h(z) = e^{-k|z|^2}$ for any positive integer k. The role of k in this example comes from the k^{th} tensor power of the line bundle, as described in 4.0.22. Consider the k^{th} tensor power of the line bundle L. The transition functions of a tensor product of line bundles are products of the original transition functions, so the transition functions of $L^{\otimes n}$ are just the powers of the transition functions of the line bundle L. The Hermitian structure is then given locally by h_0^k. The Hilbert spaces are then

$$\mathcal{H}_k = L^2_{hol}(\mathbb{C}, e^{-k|z|^2} dz \wedge d\bar{z}). \tag{7.0.16}$$

The curvature of the k^{th} tensor power of L is given by

$$\text{curv}(L^{\otimes k}) = k \ \text{curv}(L). \tag{7.0.17}$$

This is also equivalent to the condition

$$i\text{curv}(L) = k\omega. \tag{7.0.18}$$

This gives us sets of Hilbert spaces defined by

$$\mathcal{H}_k := L^2_{hol}(M, L^k), \tag{7.0.19}$$

7.1 Quantization of $\mathbb{S}^2$

Motivation 7.1.1. The two manifolds most significant in the dispersionless limit of the Toda lattice are the sphere and the torus. The quantization of the sphere will be considered first. We will need to determine the appropriate Hilbert space which will be given by holomorphic sections of a tensor power of a line bundle. We will consider the Riemann sphere with two charts given by stereographic projection.

Motivation 7.1.2. Consider two charts on $\mathbb{S}^2$, which will be called chart 1 and chart 2. Let $\{N\}$ designate the north pole of the sphere, $(0, 0, 1)$, and $\{S\}$ designate the south pole, $(0, 0, -1)$. Chart 1 is defined on $\mathbb{S}^2 \setminus \{N\}$ and chart 2 is defined on $\mathbb{S}^2 \setminus \{S\}$. The complex coordinate in chart 1 will be denoted z and the complex coordinate in chart 2 will be denoted w. If $\mathbb{S}^2$ is identified with the one dimensional complex projective space with coordinates $[z_0, z_1]$, then

$$z := \frac{z_0}{z_1} \text{ and} \tag{7.1.3}$$

$$w := \frac{z_1}{z_0} \tag{7.1.4}$$

so that the relation between z and w becomes clear:

$$zw = 1 \tag{7.1.5}$$

This relation provides the transition function

$$w(z) = \frac{1}{z}. \tag{7.1.6}$$

Remark 7.1.7. Consider a section of a holomorphic line bundle $L \to \mathbb{S}^2$. In each chart, this must look like a 1 dimensional vector space over $\mathbb{C}$. The holomorphic functions on all of $\mathbb{S}^2$ must be constant since $\mathbb{S}^2$ is compact. Therefore, we will need to consider sections instead of functions, so that the transition functions can allow us to use more interesting objects than constant functions. These sections correspond to the tensor power $L^{\otimes 1}$. In chart 1 with coordinate z, the section is given in homogeneous coordinates by $[z_0 : z_1] \mapsto ([z_0 : z_1], \frac{z_0}{z_1}) = ([z_0 : z_1], z)$, so we can denote the section by z. The sections of the tensor powers of the line bundle are powers of the sections of the first tensor power, so the sections of the n^{th} tensor power of L are given by $1, z, z^2, \ldots, z^n$. In this chart, we see that the sections have representations as functions on the chart which have a single zero at 0 with increasing order as the tensor power increases, and a pole having the same property at infinity. The transition functions allow the representation as a function in chart 2 to be holomorphic. This allows us to determine these transition functions for $L^{\otimes n}$.

Motivation 7.1.8. In the construction of quantization, line bundles must be endowed with Hermitian metrics. These will now be constructed.

Proposition 7.1.9. *The metric on a section s is given by*

$$h(s(z), s(z)) = \int s(z)\bar{s(z)} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^{2+n}}. \tag{7.1.10}$$

Proof. Differentials transform according to the rule

$$dz = \frac{-1}{w^2} dw \tag{7.1.11}$$

on the Riemann sphere. The sections will also transform in switching between charts. Consider a holomorphic section in chart 1 of the form $s_1(z) = z^k$. Holomorphicity requires that $k \geq 0$. Suppose that in the second chart, this section is written $s_2(w)$. There is an n determined by the transition function so that

$$s_2(w) = w^n s_1(\frac{1}{w}) = w^n (\frac{1}{w})^k = w^{n-k}, \tag{7.1.12}$$

which forces $n > k \geq 0$ so that the section is holomorphic in chart 2. The metric can be rewritten in the w coordinate. We will switch the section, so that if (7.1.10) is written in $s_1(z)$, we need to write the metric in $s_2(w)$. We also need to note that on the intersection of the two charts (which is all but a set of measure zero), $1 + |z|^2 = 1 + |w|^2$. The sections are related by (7.1.12), so the metric in the second chart has the form, after a short calculation,

$$h(s(w), s(w)) = \int s(w)\bar{s}(w) \frac{dw \wedge d\bar{w}}{(1 + |w|^2)^{2+n}}. \tag{7.1.13}$$

□

Remark 7.1.14. In local coordinates $w_i = \frac{z_i}{z_0}$ for $i = 1, \ldots, N$, the Fubini-Study fundamental form for projective space is defined as

$$\omega_{FS} = i \frac{(1 + |w|^2) \sum_{i=1}^{N} dw_i \wedge d\bar{w}_i - \sum_{i,j=1}^{N} \bar{w}_i w_j dw_i \wedge d\bar{w}_j}{(1 + |w|^2)^2}. \tag{7.1.15}$$

For $\mathbb{CP}^1$, this specializes to

$$\omega = \frac{i}{(1 + |w|^2)^2} dw \wedge d\bar{w}. \tag{7.1.16}$$

This form is, up to a coefficient of i, the same measure we found in the above proposition in the case of $n = 0$.

Motivation 7.1.17. Since we have found a metric for this Hilbert space, we will next find an orthonormal basis.

Proposition 7.1.18. *The dimension of the space of holomorphic sections of $L^{\otimes n}$ is n.*

The proof is an immediate consequence of the Riemann-Roch theorem. The statement of this theorem will require a few definitions.

Definition 7.1.19. Consider a 1-dimensional complex manifold. A *divisor* is a formal sum of points of the manifold with integer coefficients. Suppose f is a meromorphic function on M. If f has a zero at a point z_α, let s_α denote the order of that zero. If f has a pole at a point z_β, let $-s_\beta$ denote the order of that pole. If R is the set of all zeros and poles of f, the divisor of f is denoted

$$(f) := \sum_{z_i \in R} s_i z_i. \tag{7.1.20}$$

The divisor of a 1-form is defined similarly.

Definition 7.1.21. A divisor $D = \sum_j k_j p_j, k_j \in \mathbb{Z}$ is called positive (and represented by the notation $D \geq 0$) if $k_j \geq 0\ \forall j$.

Definition 7.1.22. We introduce two notations for a meromorphic divisor D:

$$\mathcal{L}(-D) = \{f | f \text{ is a meromorphic function and } \mathrm{ord}_f(p_i) \geq -k_i\ \forall i\} \tag{7.1.23}$$

$$\Omega(D) = \{\omega | \omega \text{ is a meromorphic differential and } (\omega) - D \geq 0\} \tag{7.1.24}$$

Theorem 7.1.25 (The Riemann-Roch Theorem)**.** *Consider a compact 1 dimensional complex manifold (also called a* Riemann surface*), M of genus g and D is a meromorphic differential.*

$$dim(\mathcal{L}(-D)) - dim(\Omega(D)) = deg(D) - g + 1 \tag{7.1.26}$$

Proof of the proposition. This result is an immediate consequence of the Riemann-Roch theorem, since the Riemann sphere has no holomorphic differentials. We are looking for holomorphic sections, so we need to consider $D = 0$. $\mathcal{L}$ is counting holomorphic sections and Ω is counting holomorphic differentials, of which there are none. This tells us that $\dim(\mathcal{L}(0)) = 1$. If each copy of L has a one dimensional space of holomorphic sections, then $L^{\otimes n}$ has an n dimensional space of holomorphic sections. This can be seen because the transition functions of $L^{\otimes n}$ are the n^{th} powers of transition functions of L. The specific case of the sphere will be worked out below. □

Proposition 7.1.27. *A basis for the Hilbert space of holomorphic sections of $L^{\otimes n}$ is given by*

$$s_k = \sqrt{\frac{(n+1)}{2\pi}\binom{n}{k}}\, z^k \tag{7.1.28}$$

for $k = 1, \ldots, n$.

Proof. Computation. □

This basis will be important in the section on Toeplitz operators. To summarize, the Hilbert space for the quantization of $\mathbb{S}^2$ is equivalent to the space

$$\mathcal{H}_N^S = \left\{ f : \mathbb{C} \to \mathbb{C} \;\middle|\; f \text{ is entire and } \frac{i}{2}\int_{\mathbb{C}} |f(z)|^2 \frac{dz \wedge d\bar{z}}{(1+|z|^2)^{N+2}} < \infty \right\}. \tag{7.1.29}$$

This identification is achieved through stereographic projection (by considering the sphere to be the completion of $\mathbb{C}$). These functions are just homogeneous polynomials with degree less than or equal to N. Recall that the symplectic form on the sphere is just an area form.

Remark 7.1.30. Recall that the quantization condition states that the integral of ω over any closed 2-dimensional surface in M be integer multiple of $2\pi\hbar$. That condition in this specific case requires that we use the above measure, $L^2(\frac{1}{(1+|z|^2)^{N+2}})$.

Proposition 7.1.31. *The transition functions for the N^{th} tensor power of a line bundle over the sphere of radius 1 are the N^{th} powers of the transition functions of the 1^{st} tensor power.*

Proof. The proof will follow much of the above work, but we will demonstrate calculation by considering the sphere in $\mathbb{R}^3$. On the sphere of radius 1, $\mathbb{S}^2$, we can introduce the form

$$\omega_N = \frac{Nh}{4\pi}(x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy). \tag{7.1.32}$$

The area is then Nh, as desired. We now want a line bundle L_N with a connection ∇ so that the form $\Omega = h^{-1}\omega_N$ is the curvature form. We will do this computation by using stereographic projection. We will again denote the north pole, $(0, 0, 1)$ by $\{N\}$ and the south pole $(0, 0, -1)$ by $\{S\}$. On the chart $U_+ = \mathbb{S}^2 \setminus \{N\}$ we introduce the complex coordinate

$$z = \frac{x_1 - ix_2}{1 + x_3} \tag{7.1.33}$$

and on $U_- = \mathbb{S}^2 \setminus \{S\}$ we introduce the complex coordinate

$$w = \frac{x_1 + ix_2}{1 - x_3}. \tag{7.1.34}$$

On the intersection we again have the relation $zw = 1$. We now have

$$\begin{cases} \Omega_N = \frac{N}{2\pi i}\frac{d\bar{z}\wedge dz}{(1+|z|^2)^2} \\ \Omega_N = \frac{N}{2\pi i}\frac{d\bar{w}\wedge dw}{(1+|w|^2)^2}. \end{cases} \tag{7.1.35}$$

We need a 1-form θ so that $d\theta = \Omega$, so we take

$$\theta_+ = \frac{Nh}{2\pi i}\frac{\bar{z}dz}{1 + |z|^2}\theta_- = \frac{Nh}{2\pi i}\frac{\bar{w}dw}{1 + |w|^2} \tag{7.1.36}$$

If we have a 1-form θ_α in a chart, then we can define a connection ∇ so that for a vector field X,

$$\nabla_X \phi = X\phi - \frac{2\pi i}{h}\theta_\alpha(X)\phi. \tag{7.1.37}$$

Notice that we can identify of the space of sections with the space of functions on the chart. On the intersection of charts $U_\alpha \cap U_\beta$, if $c_{\alpha\beta}$ denotes the transition function of sections (so $s_\alpha = c_{\alpha\beta}s_\beta$), we have

$$\theta_\beta - \theta_\alpha = \frac{h}{2\pi i} d\ \ln(c_{\alpha\beta}). \tag{7.1.38}$$

Therefore,

$$d\ \ln(c_{\alpha\beta}) = \frac{2\pi i}{h}(\theta_- - \theta_+) = N\left(\frac{\bar{w}dw}{1 + |w|^2} - \frac{\bar{z}dz}{1 + |z|^2}\right) = N\frac{dw}{w}. \tag{7.1.39}$$

This equation can be solved for the transition functions, $c_{\alpha\beta}$:

$$c_{\alpha\beta} = (w)^N = (z)^{-N}. \tag{7.1.40}$$

□

7.2 Quantization of $\mathbb{T}^2$

Motivation 7.2.1. As with the sphere, we need to find a natural setting to investigate sections of line bundles over the torus. The functions on $\mathbb{T}^2$ are equivalent to doubly periodic functions on the plane. This can be seen by considering the fundamental domain $[0,1] \times [0,1]$. A doubly periodic function on the fundamental domain determines a function on the torus by the natural projection. Since these are equivalent, we will consider the doubly periodic functions. The area form on the torus will then take a much simpler form than the area form on $\mathbb{S}^2$ did, namely

$$\omega_N = Nh\ dx \wedge dy \bmod \mathbb{Z} \times \mathbb{Z} \tag{7.2.2}$$

The map of the unit square to $\mathbb{T}^2$ can then be given by

$$\phi(x, y) = (x, y) \bmod \mathbb{Z} \times \mathbb{Z}, \tag{7.2.3}$$

so that the pullback is

$$\phi^*(\omega_N) = Nh\ dx \wedge dy. \tag{7.2.4}$$

To satisfy the quantization condition, we need a 1-form θ such that $d\theta = h^{-1}\omega$, so we will use

$$\phi^*(\theta) = Nhx\ dy \tag{7.2.5}$$

to achieve this given our maps. We will relate the sections with another set of functions, the *theta functions*. Consider the set of functions $f(z)$ satisfying the condition

$$f(z + m + in) = e^{N\pi(n^2 - 2inz)} f(z) \tag{7.2.6}$$

for a fixed value of N.

Definition 7.2.7. The set of functions that satisfy the condition (7.2.6) will be denoted Θ_N. These will be called the theta functions of order N.

Proposition 7.2.8. *The set of functions Θ_N has dimension N.*

Proof. Consider $f(z) \in \Theta_N$. We can decompose $f(z)$ in its Fourier series as $f(z) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi ikz}$. Next (7.2.6) is rewritten.

$$f(z+m+in) = e^{N\pi(n^2-2inz)} f(z) \tag{7.2.9}$$

$$\sum_{k=-\infty}^{\infty} a_k e^{2\pi ik(z+in)} = e^{N\pi(n^2-2inz)} \sum_{k=-\infty}^{\infty} a_k e^{2\pi ikz} \tag{7.2.10}$$

$$\sum_{k=-\infty}^{\infty} a_k e^{2\pi ikz-2\pi nk} = \sum_{k=-\infty}^{\infty} a_k e^{N\pi n^2-2\pi iNnz+2\pi ikz} \tag{7.2.11}$$

$$\sum_{k=-\infty}^{\infty} a_k e^{-2\pi nk} = \sum_{k=-\infty}^{\infty} a_k e^{N\pi n^2-2\pi iNnz} \tag{7.2.12}$$

$$\sum_{k-Nn=-\infty}^{\infty} a_{k-Nn} e^{-2\pi n(k-Nn)} = \sum_{k=-\infty}^{\infty} a_k e^{N\pi n^2-2\pi iNnz} \tag{7.2.13}$$

$$\sum_{k-Nn=-\infty}^{\infty} a_{k-Nn} e^{-2\pi nk+\pi Nn^2} = \sum_{k=-\infty}^{\infty} a_k e^{-2\pi iNnz} \tag{7.2.14}$$

So,

$$a_{k-Nn} = a_k e^{2\pi nk-\pi Nn^2-2\pi iNnz} = a_k e^{2\pi nk-\pi Nn^2}, \tag{7.2.15}$$

or, equivalently,

$$a_{k+Nn} = a_k e^{-(2\pi nk+\pi iNn^2)}. \tag{7.2.16}$$

In particular, the coefficients a_0 through a_{N-1} determine the entire Fourier series, so Θ is N-dimensional. □

Proposition 7.2.17. *The functions θ_j, defined by the Fourier series*

$$\theta_j = \sum_{k=-\infty}^{\infty} e^{-\pi(Nk^2+2jk)} e^{2\pi izj+2\pi iNk} \tag{7.2.18}$$

for $j = 0, \ldots, N-1$ *define an orthogonal basis for* Θ_N *with inner product*

$$h(f,g) = \int_{[0,1]\times[0,1]} f(z)\bar{g}(z)e^{-2N\pi y^2}dxdy. \tag{7.2.19}$$

Proof. Computation. Significant in this computation is the fact that

$$||\theta_j||^2 = \frac{e^{2\pi j^2/N}}{\sqrt{2N}}. \tag{7.2.20}$$

This allows us to construct an orthonormal basis. In particular, we will return to this in (8.1.5) to discuss this basis further. □

Remark 7.2.21. The quanization of Kähler manifolds is developed further in [Bor00] and [Kir90].

Chapter 8

Toeplitz Operators

Motivation 8.0.1. Toeplitz operators will be the tool used to estimate solutions to the Toda lattice. We'll first remind the reader of the conditions we impose on the symplectic manifold. M is a Kähler manifold with symplectic form ω. M must be quantizable, so $\frac{\omega}{2\pi\hbar}$ must be integral. L is a holomorphic Hermitian line bundle over M with connection ∇ with curvature form equal to ω.

Definition 8.0.2. Define the Hilbert spaces

$$\mathcal{H}_N = H^0(M, L^{\otimes N}) \tag{8.0.3}$$

for each N. These are the spaces of holomorphic sections of the N^{th} tensor power of L.

Remark 8.0.4. Recalling (7.0.17) and (7.0.18), we could also define $\mathcal{H}_N$ by the space of holomorphic sections of a Hermitian line bundle with curvature $N\omega$. The inner product on $\mathcal{H}_N$ is given by

$$\langle \psi, \phi \rangle = \int_M \langle \psi(x), \phi(x) \rangle d\lambda_x \tag{8.0.5}$$

where $d\lambda_x$ is the Liouville measure, ω^n. n is again half the dimension of the manifold. In the case of the sphere and torus, $n = 1$, so the measure will just be ω. We need to now consider a specific type of operator on $\mathcal{H}_N$.

Definition 8.0.6. We will denote orthogonal projection by

$$\Pi : L^2(M, L^{\otimes N}) \to \mathcal{H}_N. \tag{8.0.7}$$

Remark 8.0.8. So that we can discuss the case of the torus and sphere simultaneously, we will denote the Hilbert space on the sphere by $\mathcal{H}_N^S$ and the Hilbert space on the torus by $\mathcal{H}_N^T$. For the sphere, the Hilbert space $\mathcal{H}_N^S$ is a closed subspace of the space $L^2(\mathbb{C}, \frac{i}{2}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^{N+2}})$. For $\mathbb{T}^2$, the Hilbert space $\mathcal{H}_N^T$ is a closed subspace of $L^2([0,1]^2, e^{-2N\pi y^2}dx\wedge dy)$. Having defined the basis for $\mathcal{H}_N^S$, the projection is naturally defined. For $\mathbb{T}^2$, $\mathcal{H}_N^T$ is a closed subspace of $L^2([0,1]\times[0,1])$, and so there is again a natural projection.

Definition 8.0.9. A *Toeplitz operator* T is a sequence of operators $T_H^{(N)}$ acting on $\mathcal{H}_N$ which has an asymptotic expansion of the form

$$T_H^{(N)} \sim \sum_{j=0}^{\infty} N^{-j} T_{H_j}^{(N)}, \tag{8.0.10}$$

where H_j are functions such that

$$T_H^{(N)} : \mathcal{H}_N \to \mathcal{H}_N \tag{8.0.11}$$

$$f \to \Pi_N(fH). \tag{8.0.12}$$

Example 8.0.13. To understand the Toeplitz operator better, we will consider a simple example. Our functions will be functions on the circle, so we will consider the Fourier series of these functions. The space on which we will be working ($\mathcal{H}_N$ above) will be the span of $1, e^{ix}, e^{2ix}, \dots, e^{Nix}$. We would like to understand $T_H(f)$, which we will find by considering the series $T_H^{(N)}(f)$. First, we can project f onto the span of $1, e^{ix}, e^{2ix}, \dots, e^{Nix}$, producing

$$\Pi(f) = \langle f, 1\rangle \cdot 1 + \langle f, e^{ix}\rangle \cdot e^{ix} + \ldots + \langle f, e^{Nix}\rangle \cdot e^{Nix}. \tag{8.0.14}$$

To keep track of these coefficients, we set $c_0 = \langle f, 1\rangle$, $c_1 = \langle f, e^{ix}\rangle, \dots, c_N = \langle f, e^{Nix}\rangle$. We now need to multiply $\Pi(f)$ by $H = \sum_{m=-\infty}^{\infty} k_m e^{imx}$. The result is then projected back to the span of $1, e^{ix}, e^{2ix}, \dots, e^{Nix}$. We can represent this entire process by considering a matrix. The matrix of $T_H^{(N)}$ will be an $N\times N$ block of a doubly infinite matrix which is operating on the Fourier series of the function f. In our example, this matrix has the form

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ \cdots & 0 & k_0 & k_{-1} & k_{-2} & \dots & k_{-N+1} & 0 & \cdots \\ \cdots & 0 & k_1 & k_0 & k_{-1} & \dots & k_{-N+2} & 0 & \cdots \\ \cdots & 0 & k_2 & k_1 & k_0 & \dots & k_{-N+3} & 0 & \cdots \\ \cdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & 0 & k_{N-1} & k_{N-2} & \cdots & k_1 & k_0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \end{pmatrix}. \tag{8.0.15}$$

The Toeplitz operator is then defined by this sequence.

Motivation 8.0.16. Toeplitz operators can arise when we try to do operations on holomorphic sections of a line bundle. If this section is multiplied by a differentiable function, the resulting section is only differentiable, so one needs to project back to the holomorphic sections. A great deal of information will be lost in this process, we will want to consider expansions as described above. If T is considered as a map from smooth functions to endomorphisms by $f \mapsto T_f = \Pi \circ M_f \circ \Pi$, the commutative algebra of functions is being mapped to a finite-dimensional and non-commutative matrix algebra. By considering tensor powers of the line bundle (and hence larger dimensional matrix algebras), the family should in some appropriate sense approximate the algebra of functions.

Remark 8.0.17. Because of our definition in terms of having an asymptotic expansion, and since $\mathcal{H}_N$ is a finite dimensional space, being a Toeplitz operator is only seen in the limit $N \to \infty$. The order of a Toeplitz operator is defined as the first $j \geq 0$ for which $H_j \neq 0$. These operators have a structure which resembles polynomials. The composition of two Toeplitz operators is a Toeplitz operator with order equal to the sum of the orders of the summands, given in [BdMG81]. The commutator is also a Toeplitz operator with order one less than the sum of orders of the summands. We will be approximating solutions with Toeplitz operators, so we will need to know how to measure the norm of a Toeplitz operator. This is given by the following proposition.

Proposition 8.0.18. *If $T = (T^{(N)})$ is a Toeplitz operator of order m, then*

$$||T^{(N)}||_{HS} \sim d_N N^{-m} ||H_m||_2, \tag{8.0.19}$$

where $||T^{(N)}||_{HS}$ is the Hilbert-Schmidt norm and d_N is the dimension of $\mathcal{H}_N$. See [BdMG81] for the proof.

Definition 8.0.20. If a Toeplitz operator T has the expansion $\sum_{j=0}^{\infty} \Pi_N M_{H_j} \Pi_N$, then H_0 is called the *principal symbol* of T.

8.1 Toeplitz Quantization of $\mathbb{T}^2$ and $\mathbb{S}^2$

Motivation 8.1.1. We want to show that the matrix L_N, (2.3.4), is a Toeplitz operator on the torus. We will expand the operator, but we must first consider the principal symbol.

Lemma 8.1.2. *The principal symbol of the Toeplitz operator associated to the matrix L_N for the period Toda lattice,* (2.3.4), *is given by*

$$H(x, y) = b(x) + 2a(x)\cos(2\pi y), \tag{8.1.3}$$

where x and $2\pi y$ are the natural coordinates on the torus.

Motivation 8.1.4. We will consider the orthonormal basis for Θ_N which was given by

$$\vartheta_j = (2N)^{1/4} e^{-\pi j^2/N} \theta_j,\ j = 0, \dots, N-1 \tag{8.1.5}$$

We can now compute the Fourier coefficients of a function of the form

$$H(x, y) = u(x) + 2v(x)\cos(2\pi y). \tag{8.1.6}$$

The u and b function will eventually be equated and the v and a functions will be closely related.

Remark 8.1.7. As the motivation suggests, we will consider the Fourier coefficients of a function of the form

$$H(x,y) = u(y) + 2\,\cos(2\pi x)v(y). \tag{8.1.8}$$

in the orthonormal bases defined above. We will break the function into its two summands and consider each separately in the following two propositions, which will then be used to prove the lemma.

Proposition 8.1.9. *If $v(y) = 0$, so that $H(x,y) = u(y)$, the Toeplitz matrix for H is diagonal with the j^{th} diagonal entry given by*

$$\lambda_j^{(N)} = \sum_{m=-\infty}^{\infty} \hat{u}(m) e^{-\pi m^2/2N} e^{-2\pi i m j/N}. \tag{8.1.10}$$

Proof. We will consider the matrix with entries a_{jl}. Recalling (8.1.5), we see that

$$a_{jl}^{(N)} = \langle u\vartheta_l, \vartheta_j\rangle = \sqrt{2N} e^{-\pi[j^2+l^2]/N} \langle H\theta_l, \theta_j\rangle. \tag{8.1.11}$$

We now need to compute $\langle H\theta_l, \theta_j\rangle$.

$$\langle H\theta_l, \theta_j\rangle = \int_{[0,1]\times[0,1]} u\theta_l \bar{\theta}_j e^{-2N\pi y^2} dxdy \tag{8.1.12}$$

$$= \int_{[0,1]\times[0,1]} u(y) \sum_n e^{-\pi(Nn^2+2ln)} e^{2\pi i(x+iy)(l+Nn)} \tag{8.1.13}$$

$$\cdot \sum_n e^{-\pi(Nn^2+2jn)} e^{2\pi i(x+iy)(j+Nn)} e^{-2N\pi y^2} dxdy. \tag{8.1.14}$$

By considering the dependence on x, and the fact that the $f_k(x) = e^{2\pi i x k}$ are orthogonal on $[0,1]$, this is 0 unless $j = l$. This forces our matrix to be diagonal. In the case $j = l$,

$$\langle u(y)\theta_j, \theta_j\rangle = \sum_m e^{-2\pi[Nm^2+2mj]} \int_0^1 e^{-2N\pi y^2} e^{-4\pi y[j+mN]} u(y) dy \tag{8.1.15}$$

$$= \int_0^1 \left(e^{-2\pi[Ny^2+2yj]} \sum_m e^{-2\pi[Nm^2+2m(j+Ny)]} \right) u(y) dy \tag{8.1.16}$$

$$= \int_0^1 K_j^{(N)}(y) u(y) dy, \tag{8.1.17}$$

where $K_j^{(N)}(y)$ is defined by

$$K_j^{(N)}(y) = \left(e^{-2\pi[Ny^2+2yj]} \sum_m e^{-2\pi[Nm^2+2m(j+Ny)]} \right). \tag{8.1.18}$$

By using the Poisson summation formula,

$$K_j^{(N)}(y) = \frac{e^{2\pi j^2/N}}{\sqrt{2N}} \sum_m e^{2\pi i(y+j/N)m}. \tag{8.1.19}$$

Now,

$$\int_0^1 \frac{e^{2\pi j^2/N}}{\sqrt{2N}} \sum_m e^{2\pi i(y+j/N)m} u(y)dy = \frac{e^{2\pi j^2/N}}{\sqrt{2N}} \sum_m e^{2\pi i(j/N)m} \int_0^1 \sum_m e^{2\pi i y m} u(y)dy \tag{8.1.20}$$

$$= \frac{1}{\sqrt{2N}} \sum_m e^{2\pi i(j/N)m} \hat{u}(m) \tag{8.1.21}$$

We can now plug this back into (8.1.11):

$$a_{jj}^{(N)} = \sqrt{2N} e^{-\pi 2j^2/N} \frac{1}{\sqrt{2N}} \sum_m e^{2\pi i(j/N)m} \hat{u}(m) \tag{8.1.22}$$

$$= e^{-\pi 2j^2/N} \sum_m e^{2\pi i(j/N)m} \hat{u}(m) \tag{8.1.23}$$

$$= \sum_{m=-\infty}^{\infty} \hat{u}(m) e^{-\pi m^2/2N} e^{-2\pi i m j/N} \tag{8.1.24}$$

□

Remark 8.1.25. As $N \to \infty$, if $\hat{u}$ has compact support in m,

$$\lambda_j^{(N)} \sim \sum_m \hat{u}(m) e^{-2\pi i m j/N} = u(-j/N) \tag{8.1.26}$$

uniformly in j.

Proposition 8.1.27. *If $H(x, y)$ is of the form $2\ cos(2\pi x)v(y)$, then the Toeplitz matrix for H is 0 except for the super and sub diagonals and corners. The entries of* *the matrix are asymptotic to the entries of the matrix*

$$\begin{pmatrix} 0 & v(1-1/2N) & 0 & \cdots & v(1/2N) \\ v(1-1/2N) & 0 & v(1-3/2N) & \cdots & 0 \\ 0 & v(1-3/2N) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & v(3/2N) \\ v(1/2N) & 0 & 0 & v(3/2N) & 0 \end{pmatrix}. \tag{8.1.28}$$

Proof. Since the ideas of the proof are the same as the previous proposition, we will only sketch the computations here. We again begin by computing the entries of the matrix. This will be computed in the ϑ variables, which are again related to the *theta* coordinates by (8.1.5).

$$a_{jl}^{(N)} = \langle 2\cos(2\pi x)v(y)\vartheta_j, \vartheta_l\rangle \tag{8.1.29}$$

$$= 2\sum_{n,m} e^{-\pi[N(n^2+m^2)+2(jn+lm)]}\ \widehat{\cos(2\pi x)}$$

$$\cdot\,[N(m-n)+l-j]\int_0^1 e^{-2\pi[Ny^2+y(j+l+N(n+m))]}v(y)dy \tag{8.1.30}$$

This is 0 unless either $N(m-n)+l-j$ is ± 1. These will correspond to the super and sub diagonals respectively.

Case 1: $N(m-n)+l-j=1$. $N(m-n)+l-j=1$, so if $j \le N-2$, then $l=j+1$ and $m=n$. If $j > N-2$, then $j=N-1$, so $l=0$ and $m=n+1$. The $j \le N-2$ case corresponds to the super diagonal and $j=N-1$ corresponds to the bottom left corner.

Case 2: This is the mirror situation to case 1, which provides the sub diagonal and the top right corner.

We can now find the non-zero entries of the matrix explicitly.

Suppose $m=n$ and $l=j+1$:

$$\langle 2\cos(2\pi x)v(y)\theta_j, \theta_{j+1}\rangle = \int_0^1 \sum_n e^{-\pi[2Nn^2+2(jn+(j+1)n)]}e^{-2\pi[Ny^2+y(2j+1+2nN)]}v(y)dy \tag{8.1.31}$$

$$= \int_0^1 \Lambda_j(y)v(y)dy \tag{8.1.32}$$

where $\Lambda_j(y)$ is defined as a kernel exactly as the previous example. By now completing the square and using the Poisson summation as in the last case, we obtain the formula

$$\Lambda_j(y) = \frac{1}{\sqrt{2N}} e^{\frac{2\pi}{N}(j+\frac{1}{2})^2} \sum_m e^{2\pi i m[y+(j+\frac{1}{2})/N]} e^{-\pi m^2/2N} \tag{8.1.33}$$

We now use this in the orthonormal basis $\{\vartheta_j\}$:

$$\langle 2\,\cos(2\pi x)v(y)\vartheta_j, \vartheta_{j+1}\rangle = e^{-\pi/2N} \sum_m e^{-\pi m^2/2N} e^{-2\pi i m(j+\frac{1}{2})/N} \hat{v}(m) \tag{8.1.34}$$

Just as in the last example, we can note the behavior as $N \to \infty$.

$$a^{(N)}_{j,j+1} \sim v\left(-\frac{j+\frac{1}{2}}{N}\right) \tag{8.1.35}$$

uniformly in j as $N \to \infty$ □

Now suppose $j = N-1$, $l = 0$ and $m = n+1$. The matrix entry is now given by

$$\langle 2\,\cos(2\pi x)v(y)\theta_{N-1}, \theta_0\rangle = \int_0^1 \sum_n e^{-\pi[2Nn^2+4Nn+N-2n]} e^{-2\pi N[y^2+2y(1+n-\frac{1}{2N})]} v(y)dy \tag{8.1.36}$$

where we define, again:

$$\Lambda^{(N)}_j = \sum_n e^{-\pi[2Nn^2+4Nn+N-2n]} e^{-2\pi N[y^2+2y(1+n-\frac{1}{2N})]}. \tag{8.1.37}$$

After completing the square and using the Poisson summation formula again, we obtain

$$\Lambda^{(N)}_j = \frac{1}{\sqrt{2N}} e^{\pi[N-2-\frac{1}{2N}]} \sum_m e^{2\pi i m[y+1-\frac{1}{2N}]} e^{-\pi m^2/2N}. \tag{8.1.38}$$

Again, we return to the orthonormal basis $\{\vartheta_j\}$. This produces the relation

$$\langle 2\,\cos(2\pi x)v(y)\vartheta_{N-1}, \vartheta_0\rangle = e^{3\pi/2N} \sum_m e^{2\pi i m/2N} e^{-\pi m^2/2N} \hat{v}(m). \tag{8.1.39}$$

Again, letting $N \to \infty$,

$$a_{N-1,0} \sim v(\frac{1}{2N}) \tag{8.1.40}$$

uniformly in j. The case 2 described above is completely symmetric, and so will result in the matrix of the proposition being symmetric.

Proof of the Lemma, 8.1.2. By combining the previous two propositions, we obtain the matrix

$$\begin{pmatrix} u(1) & v(1-1/2N) & 0 & \cdots & v(1/2N) \\ v(1-1/2N) & u(1-1/N) & v(1-3/2N) & \cdots & 0 \\ 0 & v(1-3/2N) & u(1-2/N) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & v(3/2N) \\ v(1/2N) & 0 & 0 & v(3/2N) & u(1/N) \end{pmatrix}. \quad (8.1.41)$$

We now take

$$u(x) = b(x) \quad (8.1.42)$$

and

$$v(x) = a\left(x - \frac{1}{2N}\right). \quad (8.1.43)$$

We will now use the basis $\{(2N)^{1/4}e^{-\pi(N-1)^2/N}\theta_{N-1}, \ldots, (2N)^{1/4}\theta_0\}$, which is just a rearrangement of the original basis. Now with $a_j = a(\frac{j}{N})$ and $b_j = b(\frac{j}{N})$, we obtain the matrix given in (2.3.4). □

We can also see that the operator we have constructed is, in fact, a Toeplitz operator. In the matrix (8.1.41), we have functions u and v. We can find functions u_N and v_N so that $u_N \sim u + \sum_{j=1}^{\infty} N^{-j}u_j$ and $v_N \sim v + \sum_{j=1}^{\infty} N^{-j}v_j$. We then have our H_N functions which will be given by $H_N(x,y) = u_N(y) + 2\cos(2\pi x)v_N(y)$. Now consider the matrix

$$\mathcal{T}_k^{(N)} = \begin{pmatrix} b_k(1/N) & a_k(1/N) & 0 & \ldots & 0 \\ a_k(1/N) & b_k(2/N) & a_k(2/N) & \ldots & 0 \\ 0 & a_k(2/N) & b_k(3/N) & a_k(3/N)\ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0\cdots & \cdots & a_k(1-1/N) & b_k(1) & \end{pmatrix} \quad (8.1.44)$$

For the function H, we have now constructed an operator $T_{H,K}$ which has an asymptotic expansion

$$T_{H,K} = \sum_{k=0}^{K} N^{-k}\mathcal{T}_k^{(N)} + O(N^{-k+1}). \quad (8.1.45)$$

These are truncations of the operator T_N.

Motivation 8.1.46. The quantization of the sphere is very similar to the case of the torus. Recall the matrix L_N for the non-periodic Toda lattice, (2.3.6),

$$L_N(t) = \begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 \\ a_1 & b_2 & a_2 & \dots & 0 \\ 0 & a_2 & b_3 & a_3 \dots & 0 \\ \dots & \dots & & \dots & \vdots \\ 0 & \dots & & a_{N-1} & b_N \end{pmatrix}. \tag{8.1.47}$$

L_N is the matrix of a Toeplitz operator on the sphere. We also recall that $a_j = a(\frac{j}{N})$ and $b_j = b(\frac{j}{N})$ for $j = 1, \dots, N-1$. We also assume $b \in C^\infty([0,1])$ and $a \in \mathcal{A} := \left\{ a \in C^0([0,1]) \text{ such that } \dfrac{a(x)}{\sqrt{x(1-x)}} \in C^\infty([0,1]) \right\}$.

Lemma 8.1.48. *The matrices L_N above are the $T^{(N)}$ in definition 8.0.20 of a Toeplitz operator. Written as a function of h and θ, the height and angle of the sphere, the principal symbol is given by*

$$H_0(h,\theta) = b(h) + 2a(h)cos(\theta). \tag{8.1.49}$$

Remark 8.1.50. For more information on Toeplitz operators in this context, see [BMS94] and [Sch96].

Chapter 9
Dispersion

To understand dispersion, we will investigate its appearance in a particular setting. The following equation was studied by Korteweg and de Vries in 1895 to describe shallow water waves:

Definition 9.0.1 (The KdV Equation)**.**

$$u_t = uu_x + u_{xxx} \tag{9.0.2}$$

Here, $u_t = \frac{\partial u}{\partial t}$, etc. In a physical situation, we will have nontrivial coefficients, but we will not consider this. The terms on the right hand side of the equation correspond to different aspects of the wave, depending on the relation of the height of a wave to the depth of the water. If the water depth is much larger than the amplitude of the wave, we can approximate the equation by the *linear KdV equation*:

$$u_t = u_{xxx}. \tag{9.0.3}$$

For waves where the amplitude is much larger than the depth of the water, the term u_{xxx} becomes insignificant, so the wave can be modelled by the equation

$$u_t = uu_x. \tag{9.0.4}$$

The height of the wave is given by u, so u_t is the speed of the wave. In (9.0.4), the speed of the wave will depend on the height of the wave, so points with larger height with move faster. This will force a wave to topple, as shown in the picture below.

9.1 Solving equation (9.0.4)

We can check that if u is implicitly given by $u = f(x + ut)$, then this provides a solution for (9.0.4). An inspiration for this solution is to instead consider a solution

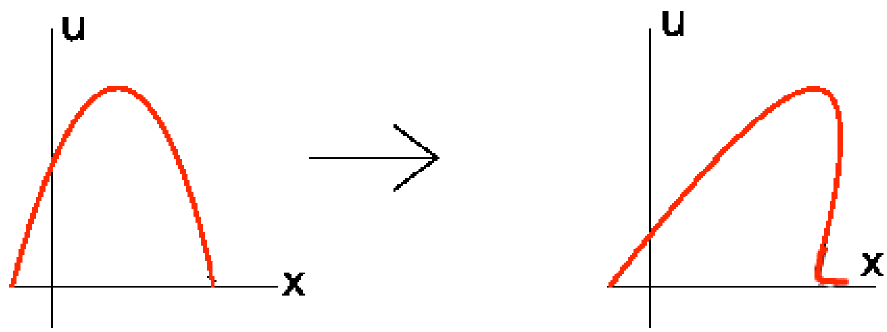


FIGURE 9.1.

to $u_t = cu_x$. We see that a travelling wave with speed c, $f(x-ct)$, is a solution. We can also see that the function f provides the solution at time 0, so the solution f is dependent on the initial conditions of the differential equation. This guess is valid as long as $\frac{\partial f}{\partial u}$ is not zero. If we look at the picture again, this condition is precisely the moment when we have a vertical tangent line to the function f, as the wave is about to topple. One reason this poses a problem is that it would signify that f is no longer a function of x.

9.2 Solving the linear KdV equation

We will next consider the situation given by (9.0.3). This equation can be solved by methods of Fourier analysis, by assuming a solution u and considering its Fourier transform, so

$$
\begin{aligned}
u_t &= u_{xxx} \\
\text{and } \hat{u}(k,t) &= \int_{-\infty}^{\infty} u(x,t)e^{ikx}dx \\
\Longrightarrow \hat{u}_t &= (-ik)^3\hat{u} \\
&= ik^3\hat{u} \\
\Longrightarrow \hat{u}(k,t) &= \hat{f}(k)e^{ik^3t} \\
\Longrightarrow u(x,t) &= \int \hat{f}(k)e^{i(kx+k^3t)}dk. \qquad (9.2.1)
\end{aligned}
$$

The function f is going to be related to u at time 0, since

$$\begin{aligned} u(x,0) &= \int \hat{f}(k)e^{ikx}dk \\ &= 2\pi f(x). \end{aligned}$$

We expect the solution to be of the form

$$\int \hat{f}(k)e^{i(kx+\omega t)}, \tag{9.2.2}$$

where $kw + \omega t$ is the phase function. k is the *wave number*, and is the spatial analogue of the wave frequency. If use this guess in (9.0.4), we see that $\omega = k^3$. $\dfrac{\omega}{k}$ is the *propagation speed* or *phase speed* and describes the speed of the phase function. This is seen by considering the initial phase function, $\theta_0 = kx + \omega t$. The position, x, is then given by $x = \dfrac{\theta_0 - \omega t}{k}$. $\frac{\omega}{k}$ is then the speed.

Definition 9.2.3. A solution which has a propagation speed dependent on the wave number is said to be *dispersive*.

The wave number k is also dependent on the wavelength λ by the relation

$$k = \frac{2\pi}{\lambda}, \tag{9.2.4}$$

so the above definition of a dispersive wave is equivalent to the requirement that the propagation speed be dependent on the wavelength. In our case, the propagation speed is given by k^2. We also see that (9.0.4) does not have this property, so (9.0.4) is also referred to as the *dispersionless KdV equation.* We will also mention one of the important solutions to the KdV equation, the soliton. This is given by

$$u(x,t) = \frac{1}{2}c^2 \operatorname{sech}^2(\frac{1}{2}c(x + c^2t)), \tag{9.2.5}$$

and this is particularly remarkable since the presence of a solitary wave solution can only be achieved by a balancing of the dispersive and nonlinear summands of the KdV equation.

Example 9.2.6 (The wave equation)**.** It can also be seen that the standard wave equation in one spacial dimension, $\frac{\partial^2 u}{\partial t^2} = c^2 u_{xx}$, is dispersionless since considering the term $e^{i(kx-\omega t)}$ will force $\omega = \pm k$, so $\frac{\omega}{k} = \pm c$, which is not dependent upon k.

While dispersionless equations can be constructed, they can also arise as quasi-classical limits of integrable systems. We can illustrate the idea here with the KdV equation. The quasi-classical limit, roughly, is obtained by taking the limit of the description of a quantum mechanical system as $\hbar \to 0$. This is a purely mathematical idea since $\hbar$ is really a constant, but the method can provide a way of relating quantum mechanics and classical mechanics.

9.3 Lax Pair Formulation of KdV

Motivation 9.3.1. The KdV equation is another example of an integrable system, and it has a Lax pair formulation, just as the Toda lattice did.

Proposition 9.3.2. *The KdV equation has a Lax pair formalism where* $\partial = \frac{\partial}{\partial x}$ *and is given by the operators*

$$L = \partial^2 + \frac{1}{6}u \tag{9.3.3}$$

$$B = 4\partial^3 + \frac{1}{2}(\partial u + u\partial) \tag{9.3.4}$$

Proof. The Lax pair structure should result in

$$\frac{\partial L}{\partial t} = [B, L]. \tag{9.3.5}$$

The KdV partial differential equation is

$$u_t = uu_x + u_{xxx}. \tag{9.3.6}$$

The rest follows by a calculation. □

Remark 9.3.7. The eigenvalue problem for this operator is significant to consider as well. The statement

$$L\psi = -\lambda\psi \tag{9.3.8}$$

becomes

$$\frac{\partial^2\psi}{\partial x^2} + \left(\frac{1}{6}u(x,t) + \lambda\right)\psi = 0, \tag{9.3.9}$$

which is precisely the time-independent Schrödinger equation. (The t variable is a parameter, not time.) The inverse scattering transform can then be used to find $u(x,t)$ given $u(x,0)$.

9.4 The WKB method

The WKB approximation method is a way to compute the quasi-classical limit. We will illustrate this by way of the time-independent Schrödinger equation,

$$\frac{-\hbar^2}{2m}\Psi'' + u\Psi = E\Psi. \tag{9.4.1}$$

Here, u is the potential energy, E is the total energy, and Ψ is the wave function. We would like to find a solution of the form $\Psi(x) = a_\hbar(x)e^{\frac{i}{\hbar}S(x)}$. We can substitute this guess into the Schrödinger equation, and by expanding S as a power series in $\hbar$, we will obtain the classical formulation (specifically, the HamiltonJacobi equation). This has the effect of removing the highest term derivative, and so in the KdV equation, we can consider the scaling

$$\frac{\partial}{\partial t} \to \alpha\frac{\partial}{\partial t} \tag{9.4.2}$$

$$\frac{\partial}{\partial x} \to \alpha\frac{\partial}{\partial x} \tag{9.4.3}$$

and take the limit $\alpha \to 0$. The KdV equation will now be

$$\alpha u_t = \alpha^3 u_{xxx} + \alpha u u_x,$$

and after taking the limit,

$$u_t = uu_x,$$

which was the dispersionless KdV equation.

Remark 9.4.4. For more on dispersionless limits, see [Bru02]. For more about solitons, see [Rem99].

Chapter 10
Bloch, Golse, Paul, and Uribe's Result

Remark 10.0.1. There are two results which Bloch et al showed recently. The two results are related to each other in that one pertains to the periodic Toda flow and one pertains to the non-periodic Toda flow. The theorems will be stated below. The reader is directed to [BGPU03] for all of the details of the proofs, which will be outlined below.

10.1 Statements of the Theorems

Theorem 10.1.1. *Let $a, b \in C^\infty(\mathbb{R})$ be 1-periodic functions, and let a_j^t and b_j^t, $j = 1, \ldots, N$ be the solution of the periodic Toda flow*

$$\dot{a}_j = a_j(b_{j+1} - b_j) \tag{10.1.2}$$

$$\dot{b}_j = 2(a_j^2 - a_{j-1}^2) \tag{10.1.3}$$

with initial conditions

$$a_j^{t=0} = a\left(\frac{j}{N}\right) \tag{10.1.4}$$

$$a_j^{t=0} = b\left(\frac{j}{N}\right). \tag{10.1.5}$$

Let us moreover suppose that there exists $s_c > 0$ such that the system

$$\partial_s a = a\partial_x b \tag{10.1.6}$$

$$\partial_s b = 2\partial_x a^2 \tag{10.1.7}$$

with initial conditions

$$a^{s=0}(x) = a(x) \tag{10.1.8}$$

$$b^{s=0}(x) = b(x) \tag{10.1.9}$$

has a smooth periodic solution for $s < s_c$. There then exist two sequences of smooth functions (determined by $a^s(x)$ and $b^s(x)$), $a^s_k(x)$ and $b^s_k(x)$, $k = 1, 2, \ldots,$ defined on $[0, s_c) \times \mathbb{R}$ and periodic in x, such that for all integers $K > 0$ and for each $\epsilon > 0$, there exist $C_k > 0$ such that for $t \leq N(s_c - \epsilon)$,

$$\forall j = 1, \ldots, N, \; \left| a^t_j - \left(a^{\frac{t}{N}}(\frac{j}{N}) + \sum_{k=1}^{K-1} N^{-k} a_k^{\frac{t}{N}}(\frac{j}{N}) \right) \right| \leq C_K N^{-K} \tag{10.1.10}$$

and

$$\forall j = 1, \ldots, N, \; \left| b^t_j - \left(b^{\frac{t}{N}}(\frac{j}{N}) + \sum_{k=1}^{K-1} N^{-k} b_k^{\frac{t}{N}}(\frac{j}{N}) \right) \right| \leq C_K N^{-K}. \tag{10.1.11}$$

In particular, as $N \to \infty$, $\frac{j}{N} \to x$, and $\frac{t}{N} \to s < s_c$,

$$a^t_j \to a^s(x) \tag{10.1.12}$$

and

$$b^t_j \to b^s(x) \tag{10.1.13}$$

where a^s and b^s are solutions of (10.1.6) *and* (10.1.7).

Theorem 10.1.14. *Let*

$$a \in \mathcal{A} = \left\{ a \in C^0([0,1]) \;\middle|\; \frac{a(x)}{\sqrt{x(1-x)}} \in C^\infty([0,1]) \right\}$$

and $b \in C^\infty(\mathbb{R})$, and let a^t_j for $j = 1, \ldots, N-1$ and b^t_j for $j = 1, \ldots, N$ be the solution of the non-periodic Toda flow

$$\dot{a}_j = a_j(b_{j+1} - b_j) \tag{10.1.15}$$

$$\dot{b}_j = 2(a_j^2 - a_{j-1}^2) \tag{10.1.16}$$

with initial conditions

$$a_j^{t=0} = a\left(\frac{j}{N}\right) \tag{10.1.17}$$

$$a_j^{t=0} = b\left(\frac{j}{N}\right). \tag{10.1.18}$$

Let us moreover suppose that there exists $s_c > 0$ such that the system

$$\partial_s a = a\partial_x b \tag{10.1.19}$$

$$\partial_s b = 2\partial_x a^2 \tag{10.1.20}$$

with initial conditions

$$a^{s=0}(x) = a(x) \tag{10.1.21}$$

$$b^{s=0}(x) = b(x) \tag{10.1.22}$$

has a solution with $a^s \in \mathcal{A}$ and $b^s \in C^\infty)[0,1])$, for $s < s_c$. Then there exist two sequences of smooth functions (determined by $a^s(x)$ and $b^s(x)$), $a_k^s(x)$ and $b_k^s(x)$, $k = 1, 2, \ldots,$ defined on $[0, s_c) \times (0,1)$ with $a_k^s(\cdot) \in \mathcal{A}$ for each $s \in [0, s_c)$, such that for all integers $K > 0$ and for each $\epsilon > 0$, there exist $C_k > 0$ such that for $t \leq N(s_c - \epsilon)$,

$$\forall j = 1, \ldots, N-1, \ \left| a_j^t - \left(a^{\frac{t}{N}}(\frac{j}{N}) + \sum_{k=1}^{K-1} N^{-k} a_k^{\frac{t}{N}}(\frac{j}{N}) \right) \right| \leq C_K N^{-K} \tag{10.1.23}$$

and

$$\forall j = 1, \ldots, N, \ \left| b_j^t - \left(b^{\frac{t}{N}}(\frac{j}{N}) + \sum_{k=1}^{K-1} N^{-k} b_k^{\frac{t}{N}}(\frac{j}{N}) \right) \right| \leq C_K N^{-K}. \tag{10.1.24}$$

In particular, as $N \to \infty$, $\frac{j}{N} \to x$, and $\frac{t}{N} \to s < s_c$,

$$a_j^t \to a^s(x) \tag{10.1.25}$$

and

$$b_j^t \to b^s(x) \tag{10.1.26}$$

where a^s and b^s are solutions of (10.1.19) *and* (10.1.20).

10.2 Sketch of the Proofs

Motivation 10.2.1. The key to the proof will lie in a lemma, stated below. This lemma will establish the existence of a Toeplitz operator which approximates the matrix L where L is either the matrix for the non-periodic or periodic Toda lattice. As will be seen in the lemma, the Lax pair structure that these matrices have will be essential.

Lemma 10.2.2. *Let $L(t)$ satisfy*

$$\frac{dL}{dt} = [L(t), B(L(t))] \tag{10.2.3}$$

where $B(L(t))$ is defined according to the whether the periodic or non-periodic lattice is being considered. We also require $L(0)$ to be given by the appropriate matrix, either (2.3.4) *or* (2.3.6)*. Let s_c be defined as in the theorem above. For all $s < s_c$, there exists a Toeplitz operator T_s such that*

$$||L(Ns) - T_s||_{HS} = O(N^{-\infty}) \tag{10.2.4}$$

where $|| \cdot ||_{HS}$ is the Hilbert Schmidt norm. Despite the change in notation, this norm is still evaluated on $\mathcal{H}_N^S$ and $\mathcal{H}_N^T$. Furthermore, for all $\epsilon > 0$, the estimates on $[0, s_c - \epsilon]$ are uniform.

Remark 10.2.5. The proof is constructive, and defines the operator in an inductive way. We are given $L(t)$ and so must construct T_s. To do this, we must find a symbol H so that T_H is "close" to L.

Fact 10.2.6. Suppose L_1 and L_2 are Toeplitz operators defined as in the Toda lattice case. This means that the operators have principal symbols given by

$$H^{1,2}(\theta, h) = a_{1,2}(h) + 2\cos(\theta)a_{1,2}(h). \tag{10.2.7}$$

It is true that $L_1 L_2$ is also a Toeplitz operator with principal symbol $H^1 H^2$ and $\frac{1}{iN}[H^1, H^2]$ is a Toeplitz operator with principal symbol given by

$$\{H_0^1, H_0^2\} = \partial_h H_0^1 \partial_\theta H_0^2 - \partial_\theta H_0^1 \partial_h H_0^2. \tag{10.2.8}$$

Remark 10.2.9. A special case of the above fact is that $\frac{1}{iN}B(L^1) = \frac{1}{iN}[L^1, \mathcal{N}]$ is a Toeplitz operator of principal symbol $\{H^1, h\} = -\partial_\theta H^1 = 2\sin(\theta)a(h)$. We now apply this to our operator T_H. If we use x to denote either h or ϕ depending on the situation, then T_H has principal symbol $H(x,\theta) = b(x) + 2a(x)cos(\theta)$ and $\frac{1}{iN}B(T_H)$ has principal symbol $-2\partial_\theta a(x)\cos(\theta)$.

Motivation 10.2.10. The next step is to construct a certain smooth one-parameter family of self-adjoint Toeplitz operators, denoted $\Lambda(t)$. We will let X denote the sphere or the torus so that the following theorem applies to either space.

Proposition 10.2.11. *Let $L_0 = \{L_0^{(N)}\}$ be a self-adjoint Toeplitz operator on X, tridiagonal in the standard basis. For notational simplicity, we will stop writing the dependence on N of the operators. Let $H_0 : X \to \mathbb{R}$ be its principal symbol. Let $J = [0, \tau]$ be a closed one-sided neighborhood of zero in $\mathbb{R}$, and assume that there exists a solution $H : J \times X \to \mathbb{R}$ of the initial value problem*

$$\begin{cases} \frac{\partial}{\partial s}H = \{H, \partial_x H\} \\ H|_{s=0} = H_0. \end{cases} \tag{10.2.12}$$

Then there exists a smooth one-parameter family of self-adjoint operators, $\Lambda(t)$, of order zero, with $\Lambda^{(N)}(t)$ defined for $\frac{t}{N} \in J$ and Toeplitz operators R and S such that

$$\begin{cases} \frac{d}{dt}\Lambda = [\Lambda, B(\Lambda)] + R \\ \Lambda|_{t=0} = L_0 + S. \end{cases} \tag{10.2.13}$$

The norms of R and S are of arbitrary order in N^{-1}. Moreover, Λ can be chosen to be tridiagonal.

Proof. The entire proof can be found in [BGPU03]. One makes increasingly accurate approximations of Λ. We begin with $\Lambda_0(s)$, which is self-adjoint and time dependent. We choose $\Lambda_0(s)$ to be an order zero Toeplitz operator with principal symbol H, making it the most general approximation we could start with. $\Lambda_0(s)$ can be chosen to be tri-diagonal, and we will see soon why this is significant. For a Toeplitz operator $\mathcal{R}_0$ of order -1,

$$\frac{d}{ds}\Lambda_0 = N[\Lambda_0, B(\Lambda_0)] + \mathcal{R}_0. \tag{10.2.14}$$

We will construct a chain of Λ_i such that the norms of R_i and S_i are of increasing order in N^{-1}. This will allow us to create Λ, R, and S of arbitrary order. Now, with $\mathcal{S}$ a Toeplitz operator of order -1 in N, define

$$\Lambda_1 = \Lambda_0 + \mathcal{S}. \tag{10.2.15}$$

We now get

$$\frac{d}{ds}\Lambda_1 = N[\Lambda_1, B(\Lambda_1)] + \mathcal{R}_1 \tag{10.2.16}$$

where $\mathcal{R}_1$ is a Toepltiz operator of order -2 as long as the symbol σ of S satisfies

$$\frac{d}{ds}\sigma = \{H, \partial_\theta \sigma\} + \{\sigma, \partial_\theta H\} - \rho_0 \tag{10.2.17}$$

where ρ_0 is the principal symbol of $\mathcal{R}_0$. This is a linearization of (10.2.12) around H. Since $\Lambda_0(s)$ was tri-diagonal, σ will also be tri-diagonal. This makes (10.2.17) solvable with smooth solutions (since it is a hyperbolic first-order $2x2$ system). Proceeding by this method, we can construct Λ_∞ so that

$$\begin{cases} \Lambda_\infty|_{s=0} = L_0 + O(N^{-\infty}) \\ \frac{d}{ds}\Lambda_\infty - N[\Lambda, [\Lambda_\infty, \mathcal{Z}]] = O(N^{-\infty}). \end{cases} \tag{10.2.18}$$

This is very nearly our desired result. We only need to make the change of variables $t = sN$, which creates the system (10.2.13), and the proof is concluded. □

Motivation 10.2.19. We now want to relate the operator Λ_∞ to the Toda lattice. This will be shown by comparing the matrices and considering an energy norm. Recall that the matrices for the Toda lattices are

$$L(t) = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & a_N \\ a_1 & b_2 & a_2 & \cdots & 0 \\ 0 & a_2 & b_3 & a_3 \cdots & 0 \\ \cdots & \cdots & & \cdots & \vdots \\ a_N & \cdots & & a_{N-1} & b_N \end{pmatrix} \tag{10.2.20}$$

for the periodic case and

$$L(t) = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & 0 \\ 0 & a_2 & b_3 & a_3 \cdots & 0 \\ \cdots & \cdots & & \cdots & \vdots \\ 0 & \cdots & & a_{N-1} & b_N \end{pmatrix} \tag{10.2.21}$$

for the non-periodic case. We will then denote the matrix of Λ_∞ by

$$\Lambda_\infty = \begin{pmatrix} B_1 & A_1 & 0 & \cdots & 0 & A_N \\ A_1 & B_2 & A_2 & 0 & \cdots & 0 \\ 0 & A_2 & B_3 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_N & \cdots & \cdots & \cdots & A_{N-1} & B_N \end{pmatrix}, \tag{10.2.22}$$

where A_N is 0 for the non-periodic lattice. We can then consider the difference,

$$L(t) - \Lambda_\infty(t) = \begin{pmatrix} \beta_1 & \alpha_1 & 0 & \cdots & 0 & \alpha_N \\ \alpha_1 & \beta_2 & \alpha_2 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \beta_3 & \alpha_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_N & 0 & \cdots & 0 & \alpha_{N-1} & \beta_N \end{pmatrix}. \tag{10.2.23}$$

Proof of 10.2.2. It is shown in [BGPU03] that we can achieve bounds on the energy function

$$E := \sum_{j=1}^{N} \left(2\alpha_j^2 + \frac{1}{2}\beta_j^2 \right) \tag{10.2.24}$$

where α_i and β_i are defined as above. Furthermore, they show that $E(t = 0) = O(N^{-\infty})$. Bloch et al go on to show by computation that $E = O(N^{-\infty})$ uniformly for $\frac{t}{N}$ bounded. Since E controls the Hilbert-Schmidt norm of $L(t) - \Lambda_\infty$, the lemma is proven. □

Proof of the Main Theorems. From 10.2.2, we know that there exists a Toeplitz operator T_s such that $||L(Ns) - T_s||_{HS} = O(N^{-\infty})$ uniformly on $[0, s_c - \epsilon]$ for every $\epsilon > 0$. We can now use the functions which were created in the matrix $\mathcal{T}_k$, 8.1.44, to approximate the true solutions to the Toda lattice. Recall

$$\mathcal{T}_k = \begin{pmatrix} b_k(1/N) & a_k(1/N) & 0 & \dots & 0 \\ a_k(1/N) & b_k(2/N) & a_k(2/N) & \dots & 0 \\ 0 & a_k(2/N) & b_k(3/N) & a_k(3/N)\dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0\cdots & \cdots & a_k(1-1/N) & b_k(1) & \end{pmatrix}. \tag{10.2.25}$$

These matrices were used to construct the Toeplitz operator $T_{H,K}$, and

$$||L(t) - T_{H,N}||_{HS} = O(N^{-(K+1)}). \tag{10.2.26}$$

We can then compare $L(t)$ with $\mathcal{T}_{k-1}$ and in particular the entries in each matrix, since we are considering a Hilbert-Schmidt norm. We can now bound the difference between exact solutions of the Toda lattice given by a_j^t and the approximation by the Toeplitz operator, giving the relations 10.1.10, 10.1.11, 10.1.23, and 10.1.24. At this point, the limits 10.1.12, 10.1.13, 10.1.25, and 10.1.26 are seen. □

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